EMPAT PENDEKATAN DALAM PEMAHAMAN TENTANG GEOMETRI

Disarikan dari :

The Four Pillars of Geometry

John Stillwell

Introduction

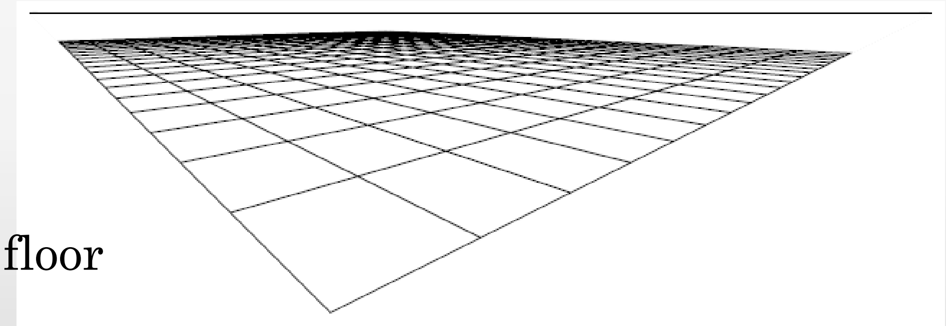
- Geometry can be developed in four fundamentally different ways, and that *all* should be used if the subject is to be shown in all its splendor.
  - *Euclid-style construction and axiomatics* √
  - *Linear algebra* √
  - *Projective geometry*
  - *Transformation groups*

- Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles. Some prefer to approach it visually, others algebraically, but the miracle is that they are all looking at the same thing.

Perspective

PREVIEW

Euclid's geometry concerns figures that can be drawn with straightedge and compass, even though many of its theorems are about straight lines alone. Are there any interesting figures that can be drawn with straightedge alone? Remember, the straightedge has no marks on it, so it is impossible to copy a length. Thus, with a straightedge alone, we cannot draw a square, an equilateral triangle, or any figure involving equal line segments. Yet there is something interesting we *can draw: a perspective view of a tiled floor, such as* the one shown in Figure 5.1.



Perspective view of a tiled floor

This picture is interesting because it seems clear that all tiles in the view are of equal size. Thus, even though we cannot draw tiles that are actually equal, we can draw tiles that *look equal.* The solution takes us into a new form of geometry—a geometry of vision—called *projective geometry.*

Perspective drawing

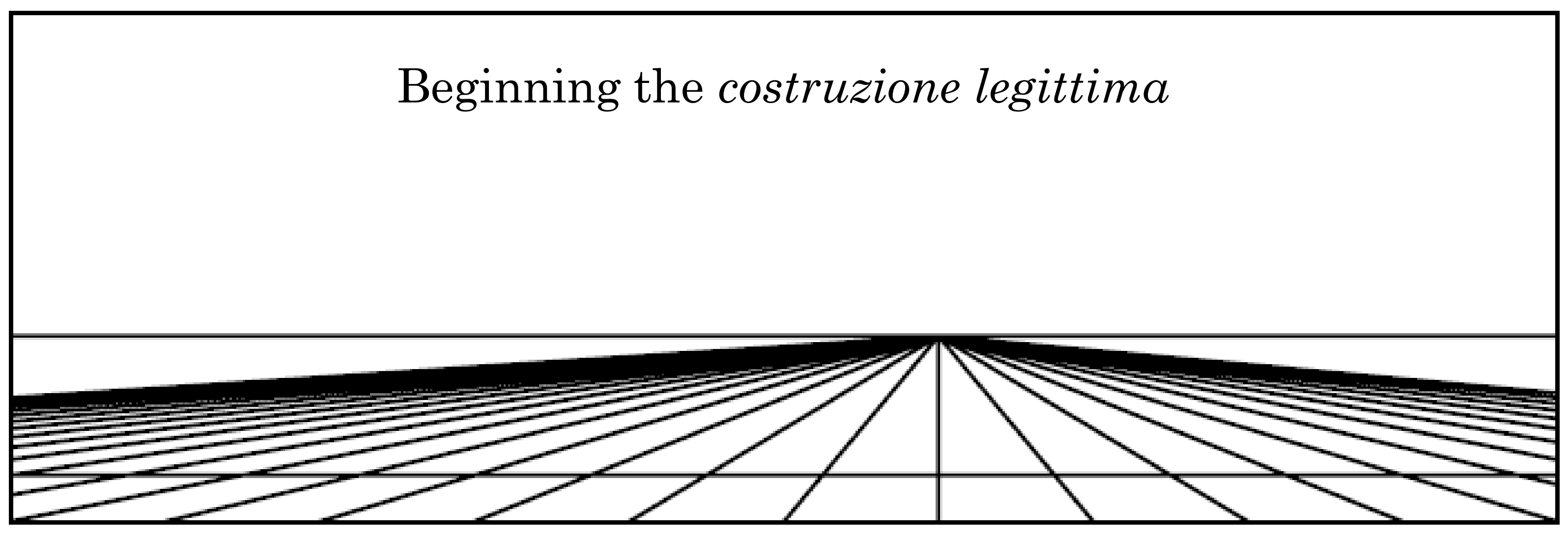
Sometime in the 15th century, Italian artists discovered how to draw three dimensional scenes in correct perspective. Figures below illustrate the great advance in realism this skill achieved, with pictures drawn before and after the discovery.

The birth of St Edmund,
by an unknown artist

St Jerome in his study,
by Albrecht D¨urer

Perspective drawing

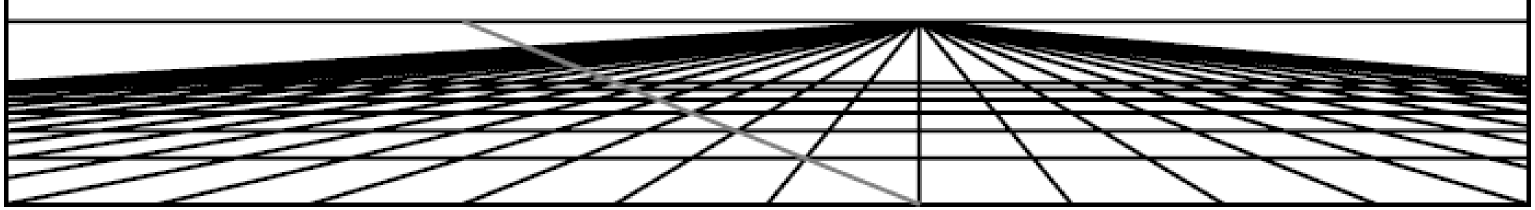
The Italians drew tiles by a method called the *costruzione legittima* (legitimate construction), first published by Leon Battista Alberti in 1436. The bottom edge of the picture coincides with a line of tile edges, and any other horizontal line is chosen as the horizon. Then lines drawn from equally spaced points on the bottom edge to a point on the horizon depict the parallel columns of tiles perpendicular to the bottom edge. Another horizontal line, near the bottom, completes the first row of tiles.



Perspective drawing

The real problem comes next. How do we find the correct horizontal lines to depict the 2nd, 3rd, 4th, . . . rows of tiles? The answer is surprisingly simple: Draw the *diagonal of any tile in the bottom row (shown in* gray in Figure below). The diagonal necessarily crosses successive columns at the corners of tiles in the 2nd, 3rd, 4th, . . . rows; hence, these rows can be constructed by drawing horizontal lines at the successive crossings.

Completing the *costruzione legittima*



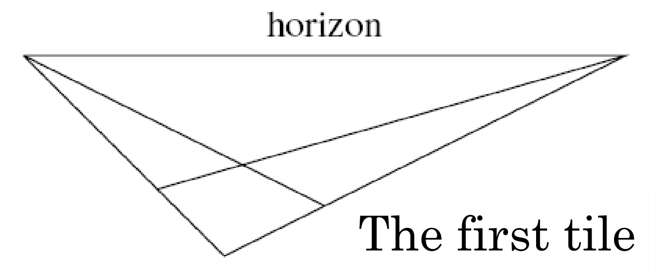
Drawing with straightedge alone

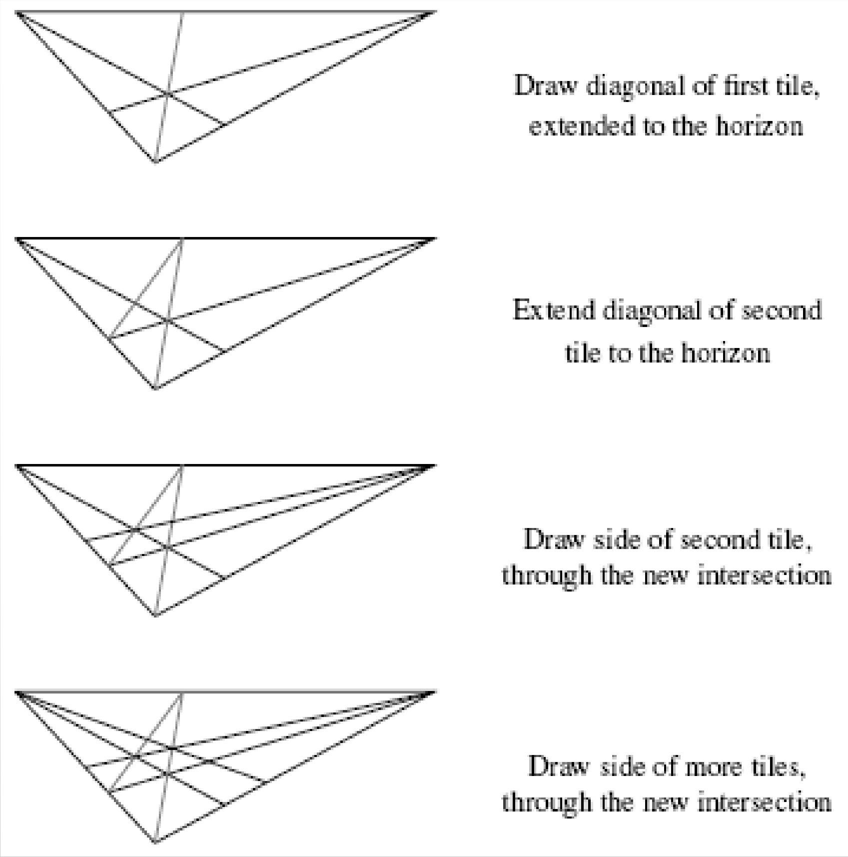
The *construzione legittima takes advantage of something that is visually* obvious but mathematically mysterious—the fact that parallel lines generally do not look parallel, but appear to meet on the horizon. The point where a family of parallels appear to meet is called their "vanishing point" by artists, and their *point at infinity by mathematicians. The horizon itself,* which consists of all the points at infinity, is called the *line at infinity.*

However, the *costruzione legittima does not take full advantage of* points at infinity. It involves some parallels that are really *drawn parallel,* so we need both straightedge and compass. The construction also needs measurement to lay out the equally spaced points on the bottom line of the picture, and this again requires a compass. Thus, the *costruzione legittima is a Euclidean construction at heart, requiring both a* straightedge and a compass.

Drawing with straightedge alone

Is it possible to draw a perspective view of a tiled floor with a straightedge alone? Absolutely! All one needs to get started is the horizon and a tile placed obliquely. The tile is created by the two pairs of parallel lines, which are simply pairs that meet on the horizon.





We then draw the diagonal of this tile and extend it to the horizon, obtaining the point at infinity of all diagonals parallel to this first one. This step allows us to draw two more diagonals, of tiles adjacent to the first one. These diagonals give us the remaining sides of the adjacent tiles, and we can then repeat the process.

Constructing the tiled floor

Projective plane axioms and their models

Drawing a tiled floor with straightedge alone requires a "horizon"—a line at infinity. Apart from this requirement, the construction works because certain things remain the same in any view of the plane:

- straight lines remain straight

- intersections remain intersections

- parallel lines remain parallel or meet on the horizon.

Now parallel lines *always meet on the horizon if you point yourself in the* right direction, so if we could look in all directions at once we would see that any two lines have a point in common. This idea leads us to believe in a structure called a *projective plane, containing objects called "points"* and *"lines" satisfying the following axioms. We write "points" and "lines"* in quotes because they may not be the same as ordinary points and lines.

Projective plane axioms and their models

Axioms for a projective plane

1. *Any two "points" are contained in a unique "line."*
2. *Any two "lines" contain a unique "point."*
3. *There exist four "points", no three of which are in a "line."*

Notice that these are axioms about *incidence: They involve only meetings* between "points" and "lines," not things such as length or angle. Some of Euclid's and Hilbert's axioms are of this kind, but not many.

- **Axiom 1 is essentially Euclid's first axiom for the construction of lines.**
- **Axiom 2 says that there are no exceptional pairs of lines that do not meet. We can define "parallels" to be lines that meet on a line called the "horizon," but this does not single out a special class of lines—in a projective plane, the "horizon" behaves the same as any other line.**
- **Axiom 3 says that a projective plane has "enough points to be interesting." We can think of the four points as the four vertices of a quadrilateral, from which one may generate the complicated structure seen in the pictures of a tiled floor at the beginning of this chapter.**

Projective plane axioms and their models

The real projective plane

If there is such a thing as a projective plane, it should certainly satisfy these axioms. But does *anything satisfy them? After all, we humans can never* see all of the horizon at once, so perhaps it is inconsistent to suppose that all parallels meet. These doubts are dispelled by the following *model, or interpretation, of the axioms for a projective plane. The model is called* the *real projective plane* RP^2*, and it gives a mathematical meaning to the* terms "point," "line," and "plane" that makes all the axioms true.

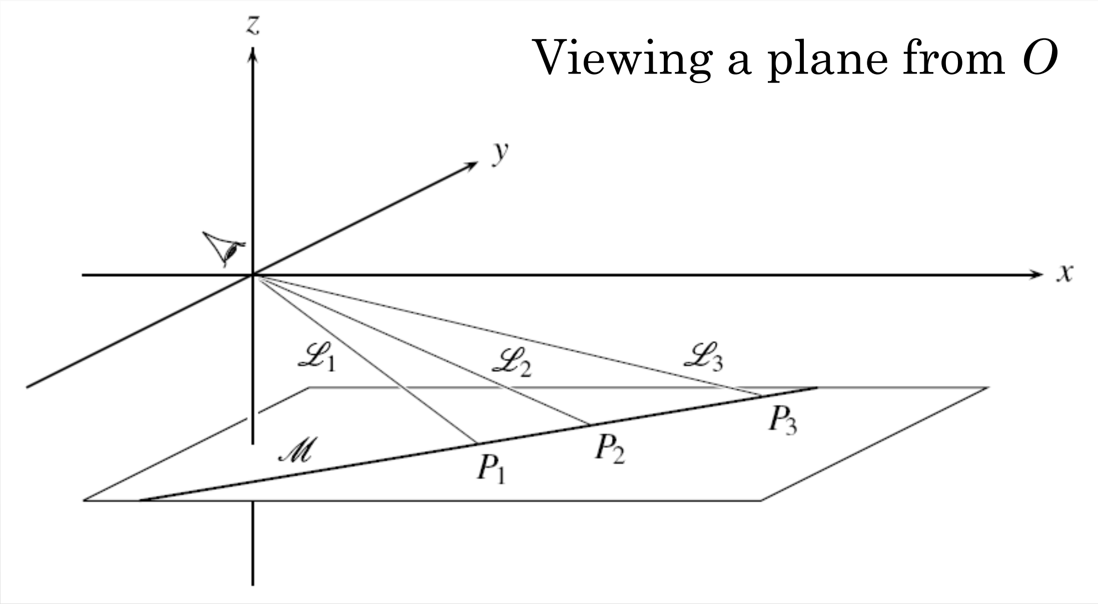
Take "points" to be lines through *0 in R3, "lines" to be planes through 0 in R3, and the "plane" to be the set of all lines through 0 in R3. Then*

1. Any two "points" are contained in a unique "line" because two given lines through *0 lie in a unique plane through 0.*
2. Any two "lines" contain a unique "point" because any two planes through *0 meet in a unique line through 0.*
3. There are four different "points," no three of which are in a "line": for example, the lines from *0 to the four points (1,0,0), (0,1,0),* (0,0,1), and (1,1,1), because no three of these lines lie in the same plane through *0.*

Projective plane axioms and their models

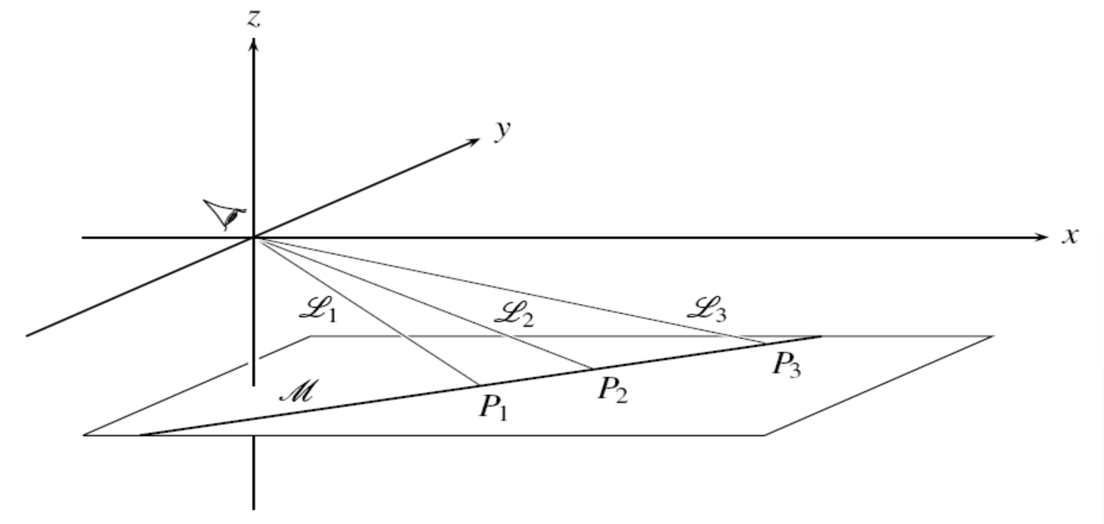
The real projective plane

It is no fluke that lines and planes through *O in R^3 behave as we want* "points" and "lines" of a projective plane to behave, because they capture the idea of *viewing with an all-seeing eye. The point O is the position of* the eye, and the lines through *O connect the eye to points in the plane.* Consider how the eye sees the plane *$z = -1$, for example (Figure below).*



Projective plane axioms and their models

The real projective plane



Points *P1,P2,P3, . . . in the plane z = −1 are joined to the eye by lines* L1,L2,L3, . . . through *O, and as the point Pn tends to infinity, the line Ln* tends toward the horizontal. Therefore, it is natural to call the horizontal lines through *O the "points at infinity" of the plane z = −1, and to call the* plane of all horizontal lines through *O the "horizon" or "line at infinity" of* the plane *z = −1.*

Unlike the lines L1,L2,L3, . . ., corresponding to points *P1,P2,P3, . . .* of the *Euclidean plane z = −1, horizontal lines through O have no counterparts* in the Euclidean plane: They *extend the Euclidean plane to a projective plane. However, the extension arises in a natural way. Once we* replace the points *P1,P2,P3, . . . by lines in space, we realize that there are* extra lines (the horizontal lines) corresponding to the points on the horizon.

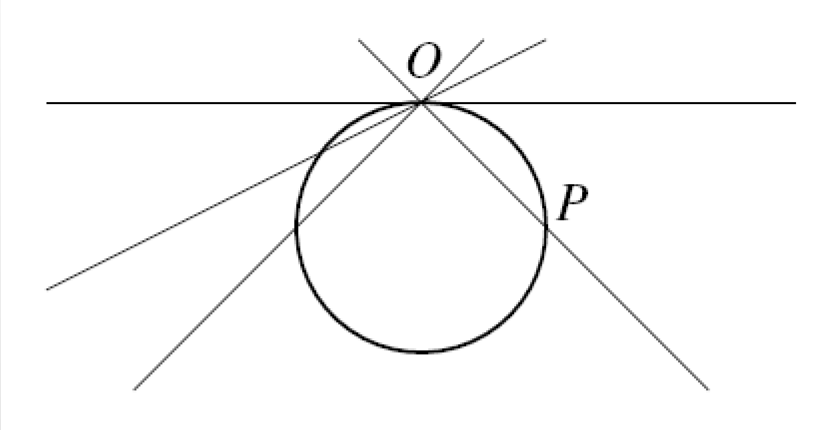
Projective plane axioms and their models

The real projective plane

This model of the projective plane nicely captures our intuitive idea of points at infinity, but it also makes the idea clearer. We can see, for example, why it is proper for each line to have only one point at infinity, not two: because the lines L connecting *O to points P along a line M in* the plane *z = −1 tend toward the same horizontal line as P tends to infinity* in either direction (namely, the parallel toM through *O).*

It is hard to find a surface that behaves like RP2, but it is easy to find a curve that behaves like any "line" in it, a so-called *real projective line. Figure 5.11 shows how. The "points" in a "line" of RP2, namely the* lines through *O in some plane through O, correspond to points of a circle* through *O. Each point P = O on the circle corresponds to the line through O and P, and the point O itself corresponds to the tangent line at O.*

Modeling a projective line by a circle



Homogeneous coordinates

Because "points" and "lines" of RP2 are lines and planes through *0 in R3*, they are easily handled by methods of linear algebra.

- **A line through *0 is* determined by any point (*x, y, z) =0, and it consists of the points (tx,ty,tz),* where *t runs through all real numbers. Thus, a "point" is not given by a* single triple (*x, y, z), but rather by any of its nonzero multiples (tx,ty,tz).* These triples are called the *homogeneous coordinates of the "point."***

- **A plane through *0 has a linear equation of the form ax+by+cz = 0,* called a *homogeneous equation. The same plane is given by the equation tax+tby+tcz=0 for any nonzero t. Thus, a "line" is likewise not given by* a single triple (*a,b, c), but by the set of all its nonzero multiples (ta,tb,tc).***

Homogeneous coordinates

It makes no algebraic difference if the coordinates of "points" and "lines" are complex numbers. We can define a *complex projective plane CP^2, each "point" of which is a set of* triples of the form (*tx,ty,tz*), *where x, y, z are particular complex numbers* and *t runs through all complex numbers. It remains true that any two* "points" lie on a unique "line" and any two "lines" have unique common point, simply because the algebraic properties of complex linear equations are exactly the same as those of real linear equations. Similarly, one can show there are four "points," no three of which are in a "line" of CP^2.

Thus, there is more than one model of the projective plane axioms. Later we shall look at other models, which enable us to see that certain properties of RP^2 are not properties of all projective planes and hence do not follow from the projective plane axioms.

Homogeneous coordinates

Projective space

It is easy to generalize homogeneous coordinates to quadruples (w, x, y, z) and hence to define the three-dimensional real projective space RP^3. It has "points," "lines," and "planes" defined as follows (we use vector notation to shorten the definitions):

- A "point" is a line through 0 in R^4, that is, a set of quadruples t**u**, where **u** = (w, x, y, z) is a particular quadruple of real numbers and t runs through all real numbers.

- A "line" is a plane through 0 in R^4, that is, a set t1**u**1 +t2**u**2 where **u**1 and **u**2 are linearly independent points of R^4 and t1 and t2 run through all real numbers.

- A "plane" is a three-dimensional space through 0 in R^4, that is, a set t1**u**1+t2**u**2+t3**u**3, where **u**1, **u**2, and **u**3 are linearly independent points of R^4 and t1, t2, and t3 run through all real numbers.

Homogeneous coordinates

Projective space

Linear algebra then enables us to show various properties of the "points," "lines," and "planes" in RP3, such as:

1. Two "points" lie on a unique "line."

2. Three "points" not on a "line" lie on a unique "plane."

3. Two "planes" have unique "line" in common.

4. Three "planes" with no common "line" have one common "point."

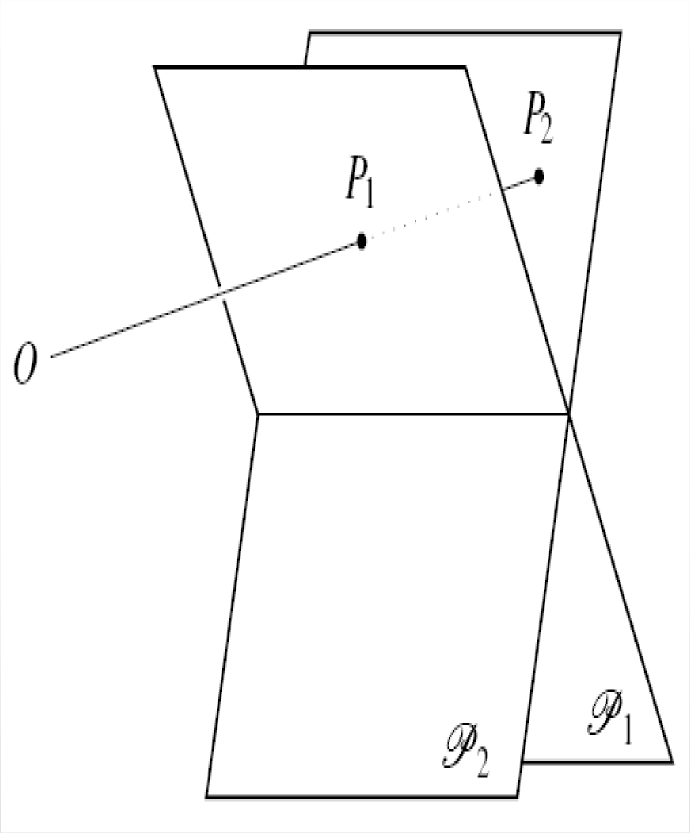
These properties hold for any *three-dimensional projective space, and RP^3* is not the only one. There is also a complex projective space CP^3, and many others. RP^3 has an unexpected influence on the geometry of the sphere.

Projection

The three-dimensional Euclidean space R^3, in which the lines through O are the "points" of RP^2 and the planes through O are the "lines" of RP^2, also contains many other planes. Each planeP not passing through O can be regarded as a perspective view of the projective plane RP^2, a view that contains all but one "line" of RP^2.

Each point P of P corresponds to a line ("of sight") through O, and hence to a "point" of RP^2. The only lines through O that do not meet P are those parallel to P, and these make up the line at infinity or horizon of P.

If P1 and P2 are any two planes not passing through O we can project P1 to P2 by sending each point P1 in P1 to the point P2 in P2 lying on the same line through O as P1. The geometry of RP^2 is called "projective" because it encapsulates the geometry of a whole family of planes related by projection.



Projecting one plane to another

Projection

Projections of projective lines

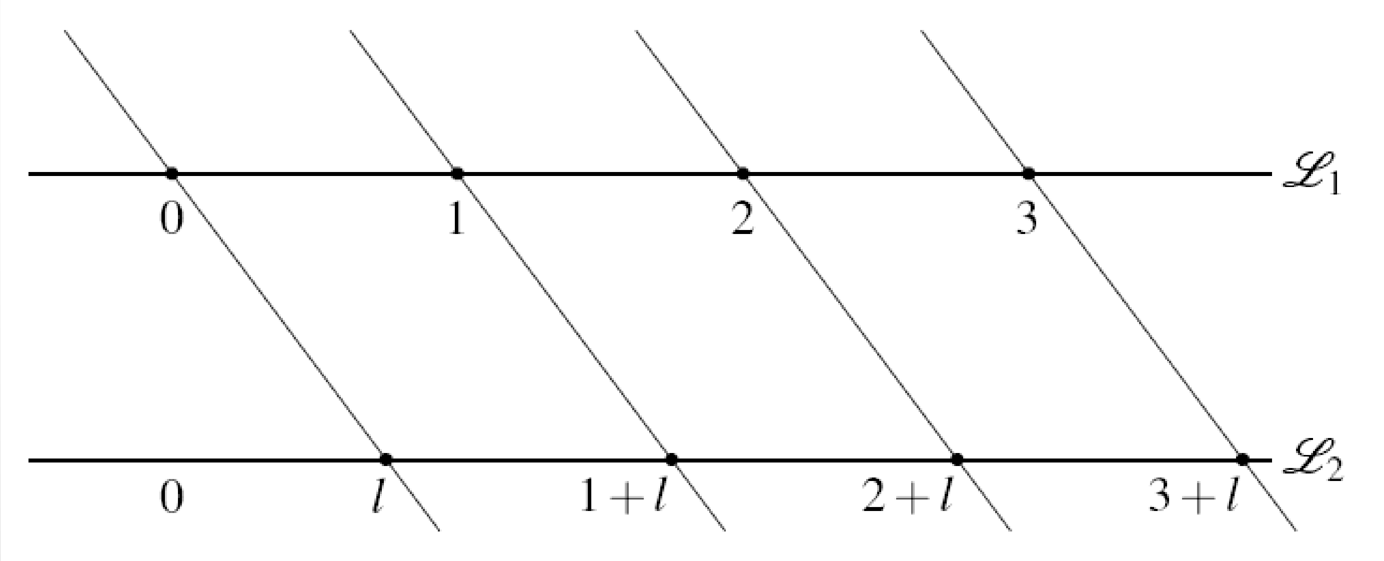
Projection of one plane P1 onto another plane P2 produces an image of P1 that is generally distorted in some way. Nevertheless, straight lines remain straight under projection, so there are limits to the amount of distortion in the image. To better understand the nature and scope of projective distortion, in this subsection we analyze the mappings of the projective line obtainable by projection.

An effective way to see the distortion produced by projection of one line L1 onto another line L2 is to mark a series of equally spaced dots on L1 and the corresponding image dots on L2. You can think of the image dots as "shadows" of the dots on L1 cast by light rays from the point of projection P, except that we have projective lines through P, not rays, so it can seem as though the "shadow" on L2 comes ahead of the dot on L1.

Projection

Projections of projective lines

In the simplest cases, where L1 and L2 are parallel, the image dots are also equally spaced. Figure 5.13 shows the case of *projection from a point at infinity, where the lines from the dots on L1 to their images on L2 are* parallel and hence the dots on L1 are simply translated a constant distance *l. If we choose an origin on each line and use the same unit of length on* each, then projection from infinity sends each *x on L1 to x+l on L2.*

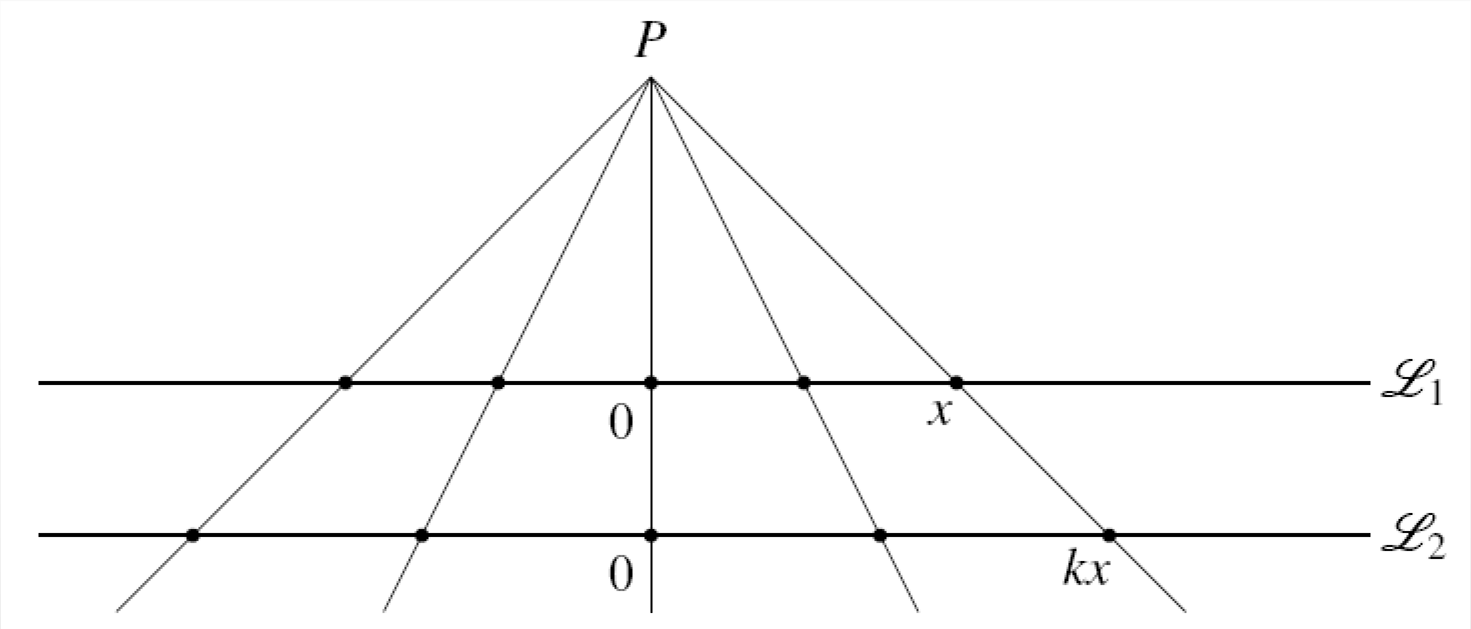


Projection from infinity

Projection

Projections of projective lines

When L1 is projected from a finite point P, then the distance between dots is magnified by a constant factor k = 0. If we take P on a line through the zero points on L1 and L2, then the projection sends each x on L1 to kx on L2. Note also that this projection sends x on L2 to x/k on L1, so the magnification factor can be any k = 0.



Projection from a finite point

Projection

Projections of projective lines

When L1 and L2 are not parallel the distortion caused by projection is more extreme. Figure A shows how the spacing of dots changes when L1 is projected onto a perpendicular line L2 from a point O equidistant from both. Figure B is a closeup of the image line L2, showing how the image dots "converge" to a point corresponding to the horizontal line through O (which corresponds to the point at infinity on L1).

Fig A. Example of projective distortion of the line

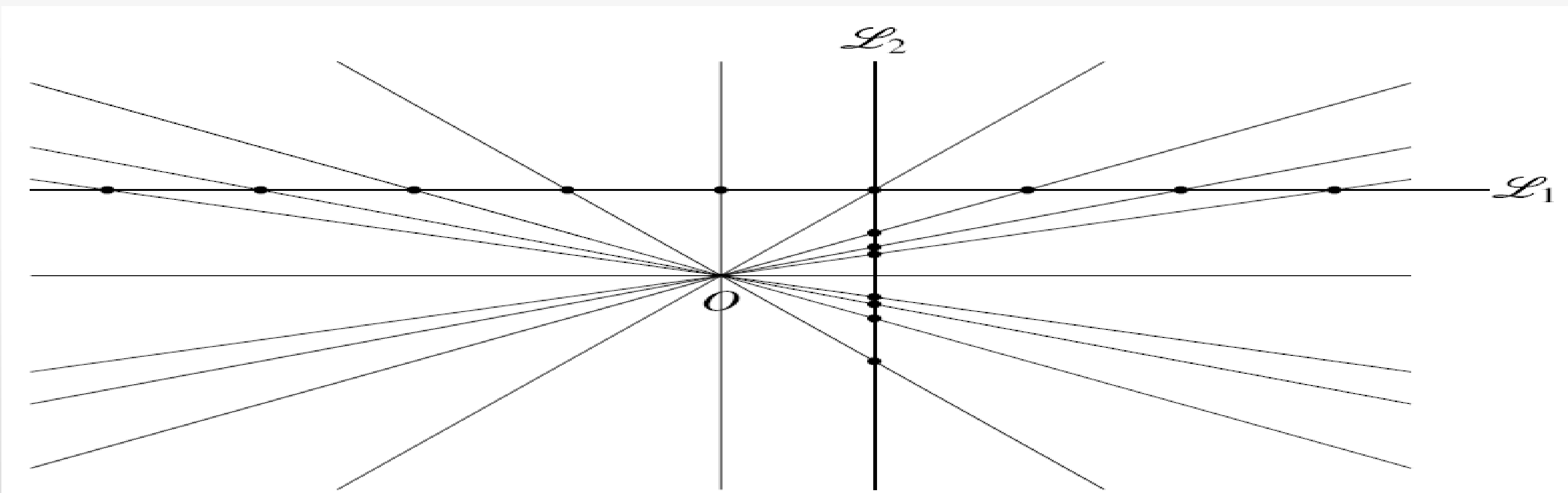
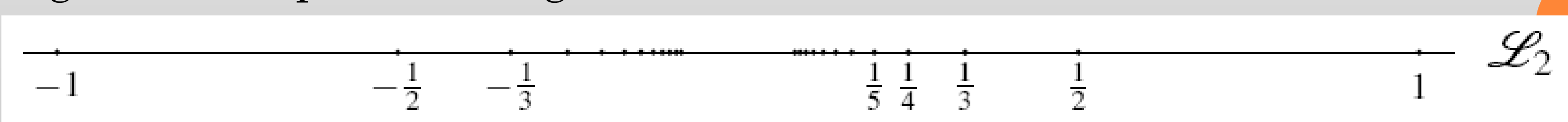


Fig. B. Closeup of the image line



Projection

Projections of projective lines

We take O = (0,0) as usual, and we suppose that L1 is parallel to the x-axis, that L2 is parallel to the y-axis, and that the dots on L1 are unit distance apart. Then the line from O to the dot at x = n on L1 has slope 1/n and hence it meets the line L2 at y = 1/n. Thus the map from L1 to L2 is the function sending x to y = 1/x. This map exhibits the most extreme kind of distortion induced by projection, with the point at infinity on L1 sent to the point y = 0 on L2.

Any combination of these projections is therefore a combination of functions 1/x, kx, and x+l, which are called generating transformations. The combinations of generating transformations are precisely the functions of the form

$$f(x) = \frac{ax+b}{cx+d}, \quad \text{where} \quad ad - bc \neq 0,$$

Linear Fractional Functions

The functions sending x to 1/x, kx, and x+l are among the functions called linear fractional, each of which has the form

$$f(x) = \frac{ax+b}{cx+d}, \quad \text{where} \quad ad - bc \neq 0,$$

The condition ad –bc = 0 ensures that f (x) is not constant. Constancy occurs only if ax+b = (a/c) (cx+d) ; in which case, ad –bc = 0 because (ad/ c) = b.

Any linear fractional function with c = 0 may be written in the form :

$$f(x) = \frac{a}{c} + \frac{bc - ad}{c(cx+d)}.$$

Such a function may therefore be composed from functions sending x to 1/x, kx, and x+l — the functions that reciprocate, multiply by k, and add l — for various values of k and l:

- first multiply x by c,
- then add d,
- then multiply again by c,
- then reciprocate,
- then multiply by bc–ad,
- and finally add ac

Thus, any linear fractional function is composed from the functions that reciprocate, multiply by k, and add l, and hence any linear fractional function on the number line is realized by a sequence of projections of the line.

Linear Fractional Functions

Dividing by zero

You remember from high-school algebra that division by zero is not a valid operation. Nevertheless, in carefully controlled situations, it is permissible, and even enlightening, to divide by zero. One such situation is in projective mappings of the projective line.

The linear fractional functions we have used to describe projective mappings of lines are actually defective if the variable x runs only through the set R of real numbers. For example, the function $f(x) = 1/x$ we used to map points of the line L1 onto points of the line L2 does not in fact map all points. It cannot send the point $x = 0$ anywhere, because $1/0$ is undefined. This defect is neatly fixed by extending the function $f(x) = 1/x$ to a new object $x = \infty$, and declaring that $1/\infty = 0$ and $1/0 = \infty$. The new object ∞ is none other than the point at infinity of the line L1, which is supposed to map to the point 0 on L2. Likewise, if $1/0 = \infty$, the point 0 on L1 is sent to the point ∞ on L2, as it should be. Thus, the function $f(x) = 1/x$ works properly, not on the real line R, but on the real projective line $R\cup\{\infty\}$—a line together with a point at infinity. The rules $1/\infty = 0$ and $1/0 = \infty$ simply reflect this fact.

It is much the same with any linear fractional function. The denominator of the fraction is 0 when $x = -d/c$, and the correct value of the function in this case is ∞. Conversely, no real value of x gives $f(x)$ the value a/c, but $x = \infty$ does. For this reason, any function $f(x) = ax+b / cx+d$ with $ad-bc =0$ maps the real projective line $R\cup\{\infty\}$ onto itself.

Linear Fractional Functions

The real projective line RP^1

We can now give an algebraic definition of the object we called the "real projective line". It is the set $R\cup\{\infty\}$ together with all the linear fractional functions mapping $R\cup\{\infty\}$ onto itself. We call this set, with these functions on it, the real projective line RP^1.

The set $R\cup\{\infty\}$ certainly has the points we require for a projective line; the functions are to give $R\cup\{\infty\}$ the "elasticity" of a line that undergoes projection. The ordinary line R is not very "elastic" in this sense. Once we have decided which point is 0 and which point is 1, the numerical value of every point on R is uniquely determined. In contrast, the position of a point on RP^1 is not determined by the positions of 0 and 1 alone.

For example, there is a projection that sends 0 to 0, 1 to 1, but 2 to 3. Nevertheless, there is a constraint on the "elasticity" of RP^1. If 0 goes to 0, 1 goes to 1, and 2 goes to 3, say, then the destination of every other point x is uniquely determined.

The Cross Ratio

It is visually obvious that projection can change lengths and even the ratio of lengths, because equal lengths often appear unequal under projection. And yet we can recognize that the first Figure in this section is a picture of equal tiles, even though they are unequal in size and shape. Some clue to their equality must be preserved, but what? It cannot be length; it cannot be a ratio of lengths; but, surprisingly, it can be a *ratio of ratios, called the cross-ratio.*

The cross-ratio is a quantity associated with four points on a line. If the four points have coordinates *p, q, r, and s, then their cross-ratio is the* function of the ordered 4-tuple (*p,q, r, s) defined by*

$$\frac{(r-p)/(s-p)}{(r-q)/(s-q)}, \quad \text{which can also be written as} \quad \frac{(r-p)(s-q)}{(r-q)(s-p)}.$$