EMPAT PENDEKATAN DALAM PEMAHAMAN TENTANG GEOMETRI

Disarikan dari :

THE FOUR PILLARS OF GEOMETRY

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Introduction

- Geometry can be developed in four fundamentally different ways, and that *all* should be used if the subject is to be shown in all its splendor.
  - *Euclid-style construction and axiomatics*
  - *Linear algebra*
  - *Projective geometry*
  - *Transformation groups*

- Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles. Some prefer to approach it visually, others algebraically, but the miracle is that they are all looking at the same thing.

Understanding Geometry through Linear Algebra

Coordinates

- Around 1630, Pierre de Fermat and Ren´e Descartes independently discovered the advantages of numbers in geometry, as *coordinates*. Descartes was the first to publish a detailed account, in his book G´eom´etrie of 1637. For this reason, he gets most of the credit for the idea and the coordinate approach to geometry became known as *Cartesian* (from the old way of writing his name: Des Cartes).

- Descartes thought that geometry was as Euclid described it, and that numbers merely *assist* in studying geometric figures. But later mathematicians discovered objects with "non-Euclidean" properties, such as "lines" having more than one "parallel" through a given point. To clarify this situation, it became desirable to *define* points, lines, length, and so on, and to *prove* that they satisfy Euclid's axioms.

Coordinates

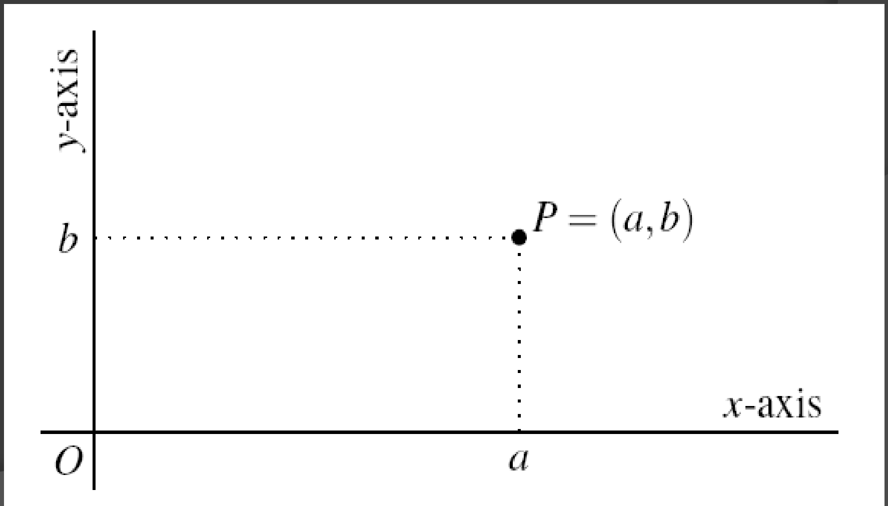
- This program, carried out with the help of coordinates, is called the *arithmetization of geometry.* In the first three sections of this chapter, we do the main steps, using the set R of real numbers to define the *Euclidean plane* R2 and the points, lines, and circles in it. We also define the concepts of distance and (briefly) angle, and show how some crucial axioms and theorems follow. However, arithmetization does much more.
  - It gives an algebraic description of constructibility by straightedge and compass, which makes it possible to prove that certain figures are *not* constructible.
  - It enables us to define what it means to “move” a geometric figure, which provides justification for Euclid’s proof of SAS, and raises a new kind of geometric question : What kinds of “motion” exist?

The number line and the number plane

- The set R of real numbers results from filling the gaps in the set Q of rational numbers with *irrational* numbers, such as √2. This innovation enables us to consider R as a *line*, because it has no gaps and the numbers in it are ordered just as we imagine points on a line to be. We say that R, together with its ordering, is a *model* of the line.

- The first step is to build the "plane," and in this we are guided by the properties of parallels in Euclid's geometry. We imagine a pair of perpendicular lines, called the *x-axis* and the *y-axis*, intersecting at a point *O* called the *origin*. We interpret the axes as number lines, with *O* the number 0 on each, and we assume that the positive direction on the *x*-axis is to the right and that the positive direction on the *y*-axis is upward.

Axes and coordinates



The number line and the number plane

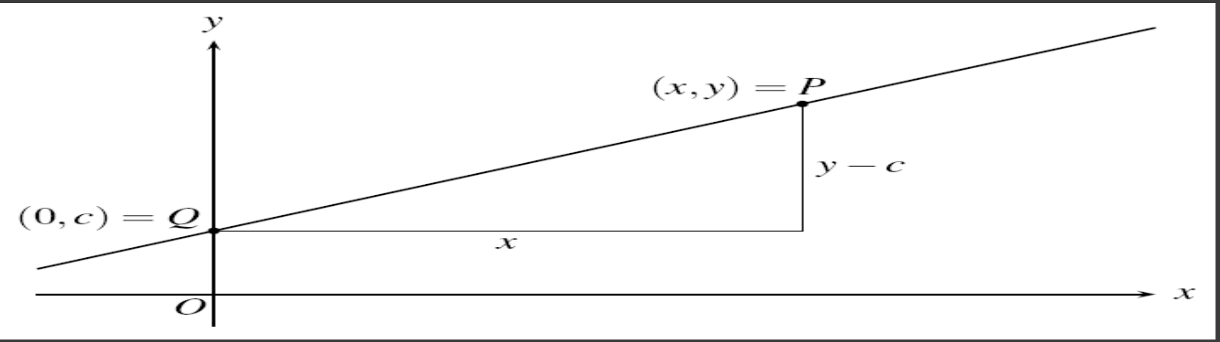
- Through any point *P*, there is (by the parallel axiom) a unique line parallel to the *y*-axis and a unique line parallel to the *x*-axis. These two lines meet the *x*-axis and *y*-axis at numbers *a* and *b* called the *x*- and *ycoordinates* of *P*, respectively. It is important to remember which number is on the *x*-axis and which is on the *y*-axis, because obviously the point with *x*-coordinate = 3 and *y*-coordinate = 4 is different from the point with *x*-coordinate =4 and *y*-coordinate =3.

- To keep the *x*-coordinate *a* and the *y*-coordinate *b* in their places, we use the *ordered pair* (*a*,*b*). For example, (3,4) is the point with *x*-coordinate =3 and *y*-coordinate =4, whereas (4,3) is the point with *x*-coordinate =4 and *y*-coordinate = 3. The ordered pair (*a*,*b*) specifies *P* uniquely because any other point will have at least one different parallel passing through it and hence will differ from *P* in either the *x*- or *y*-coordinate.

- Thus, given the existence of a *number line* R whose points are real numbers, we also have a *number plane* whose points are ordered pairs of real numbers. We often write this number plane as R×R or R^2.

Lines and their equations

- When coordinates are introduced, this allows us to define the property of straight lines known. as *slope* You know from high-school mathematics that slope is the quotient “rise over run” and, more importantly, that the value of the slope does not depend on which two points of the line define the rise and the run.



- Now suppose we are given a line of slope *a that crosses the y-axis at* the point *Q where y = c. If P = (x, y) is any point on this line,* then the rise from *Q to P is y– c and the run is x. Hence*

$$\text{slope} = a = \frac{y - c}{x}$$

and therefore, multiplying both sides by *x, y– c = ax, that is,* $y = ax + c$ *.*

- This equation is satisfied by all points on the line, and only by them, so we call it the *equation of the line*.

Lines and their equations

- Almost all lines have equations of this form; the only exceptions are lines that do not cross the *y-axis. These are the vertical lines, which also* do not have a slope as we have defined it, although we could say they have *infinite slope. Such a line has an equation of the form* $x = c$ *, for some constant c.*

- Thus, all lines have equations of the form $ax+by+c = 0$ *, for some constants a, b, and c,* called a *linear equation in the variables x and y.*

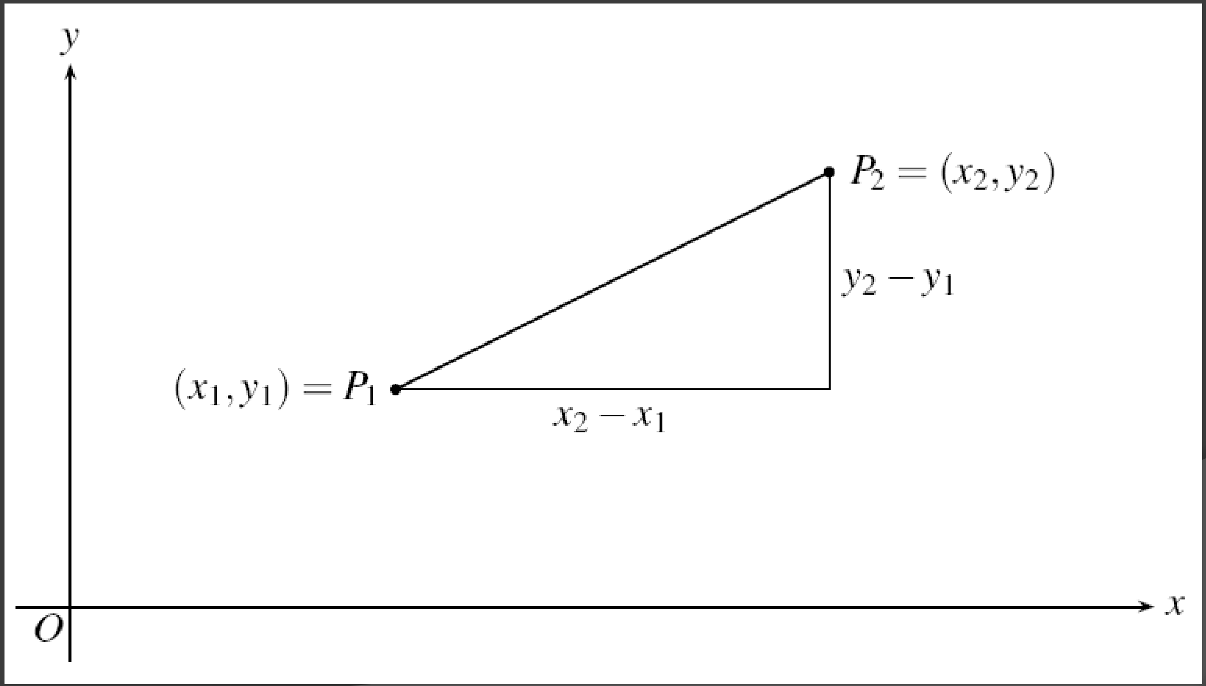
- Up to this point we have been following the steps of Descartes, who viewed equations of lines as *information deduced from Euclid's axioms (in* particular, from the parallel axiom). It is true that Euclid's axioms prompt us to describe lines by linear equations, but we can also take the opposite view: Equations *define what lines and curves are, and they provide a model* of Euclid's axioms—showing that geometry follows from properties of the real numbers.

- In particular, if a line is defined to be the set of points (*x, y) in the* number plane satisfying a linear equation then we can prove the following statements that Euclid took as axioms:
  - *there is a unique line through any two distinct points,*
  - *for any line L and point P outside L, there is a unique line through P not meeting L.*

Distance

- We introduce the concept of *distance or length into the number plane R^2* much as we introduce lines. First we see what Euclid's geometry *suggests* distance should mean; then we turn around and take the suggested meaning as a definition.

- Suppose that *P1 = (x1, y1) and P2 = (x2, y2) are any two points in R^2.* Then it follows from the meaning of coordinates that there is a right-angled triangle as shown in Figure below, and that |*P1P2| is the length of its hypotenuse. x*



Distance

- The vertical side of the triangle has length *y2 – y1, and the horizontal* side has length *x2– x1. Then it follows from the Pythagorean theorem that*

$$|P_1P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

and therefore,

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

- Thus, it is sensible to *define the distance |P1P2| between any two points P1 and P2 by the formula . If we do this, the Pythagorean theorem is* virtually "true by definition." It is certainly true when the right-angled triangle has a vertical side and a horizontal side. And we will see later how to rotate any right-angled triangle to such a position (without changing the lengths of its sides).

Distance

The equation of a circle

- The distance formula leads immediately to the equation of a circle, as follows. Suppose we have a circle with radius *r and center at the point P = (a,b). Then any point Q = (x, y) on the circle is at distance r from P,* and hence formula gives :

$$r = |PQ| = \sqrt{(x-a)^2 + (y-b)^2}.$$

Squaring both sides, we get

$$(x-a)^2 + (y-b)^2 = r^2.$$

- We call this the *equation of the circle because it is satisfied by any point* (*x, y) on the circle, and only by such points.*

Distance

The equidistant line of two points

- A circle is the set of points equidistant from a point—its center. It is also natural to ask: What is the set of points equidistant from *two points in R2?* Answer: *The set of points equidistant from two points is a line.* To see why, let the two points be *P1 = (a1,b1) and P2 = (a2,b2). Then* a point *P = (x, y) is equidistant from P1 and P2 if |PP1| = |PP2|, that is, if x* and *y satisfy the equation*

$$(x-a1)^2+(y-b1)^2=(x-a2)^2+(y-b2)^2.$$

- Squaring both sides of this equation, we get

$$(x-a1)^2+(y-b1)^2=(x-a2)^2+(y-b2)^2.$$

- Expanding the squares gives

$$x^2-2a1x+a21+y^2-2b1y+b21=x^2-2a2x+a22+y^2-2b2y+b22$$

- The important thing is that the x^2 *and* y^2 *terms now cancel, which leaves* the *linear equation*

$$2(a2-a1)x+2(b2-b1)y+(b21-b22)=0$$

- Thus, the points *P = (x, y) equidistant from P1 and P2 form a line.*

Intersections of lines and circles

- Now that lines and circles are defined by equations, we can give exact algebraic equivalents of straightedge and compass operations :

  - *Drawing a line through given points corresponds to finding the equation of the line through given points (x1, y1) and (x2, y2).*

  - *Drawing a circle with given center and radius corresponds to finding the equation of the circle with given center (a,b) and given radius r.*

  - Finding new points as intersections of previously drawn lines andcircles corresponds to finding the solution points of
    - – a pair of equations of lines,
    - – a pair of equations of circles,
    - – the equation of a line and the equation of a circle.

  Solving linear equations requires only the operations +,−,×, and ÷, and the quadratic formula shows that √ is the only additional operation needed to solve quadratic equations. Thus, all intersection points involved in a straightedge and compass construction can be found with the operations +,−,×,÷, and √.

Intersections of lines and circles

- The operations +,−,×,÷, and √ can be carried out by straightedge and compass. Hence, we get the following result:

- Algebraic criterion for constructibility. *A point is constructible (starting from the points 0 and 1) if and only if its coordinates are obtainable from the number 1 by the operations +,−,×,÷, and √.*

- The algebraic criterion for constructibility was discovered by Descartes, and its greatest virtue is that it enables us to prove that certain figures or points are *not constructible. For example, one can prove that the number* $\sqrt[3]{2}$ is not constructible by showing that it cannot be expressed by a finite number of square roots, and one can prove that the angle π/3 cannot be trisected by showing that cos π9 also cannot be expressed by a finite number of square roots. These results were not proved until the 19th century, by Pierre Wantzel. Rather sophisticated algebra is required, because one has to go beyond Descartes' concept of constructibility to survey the *totality of* constructible numbers.

Angle and slope

- The concept of distance is easy to handle in coordinate geometry because the distance between points (*x1, y1) and (x2, y2) is an algebraic function of* their coordinates. This is *not the case for the concept of angle. The angle θ between a line y = tx and the x-axis is tan−1 t, and the function tan−1 t is not an algebraic* function. Nor is its inverse function *t = tanθ or the related functions sinθ* (sine) and cosθ (cosine).

- To stay within the world of algebra, we have to work with the slope *t* rather than the angle θ . Lines make the same angle with the *x-axis if they* have the same slope, but to test equality of angles in general we need the concept of *relative slope: If line L1 has slope t1 and line L2 has slope t2,* then the *slope of L1 relative to L2 is defined to be*

$$\pm\left|\frac{t_1 - t_2}{1 + t_1 t_2}\right|$$

- The reason for the ± sign and the absolute value is that the slopes *t1, t2 alone do not specify an angle—they* specify only a pair of lines and hence a pair of angles that add to a straight angle.

- At any rate, with some care it is possible to use the concept of relative slope to test algebraically whether angles are equal. The concept also makes it possible to state the SAS and ASA axioms in coordinate geometry, and to verify that all of Euclid's and Hilbert's axioms hold. We omit the details because they are laborious, and because we can approach SAS and ASA differently now that we have coordinates. Specifically, *it becomes possible to define the concept of "motion" that Euclid appealed to in his proof of SAS! This will be done in the next section.*

Isometries

- A possible weakness of our model of the plane is that it seems to single out a particular point (the origin *0) and particular lines (the x- and y-axes).* In Euclid's plane, each point is like any other point and each line is like any other line. We can overcome the apparent bias of R^2 by considering *transformations that allow any point to become the origin and any line to* become the *x-axis. As a bonus, this idea gives meaning to the idea of* "motion" that Euclid tried to use in his attempt to prove SAS.

- A transformation of the plane is simply a function *f :R2→R2, in other* words, a function that sends points to points. A transformation *f is called an isometry (from the Greek for "same* length") if it sends any two points, *P1 and P2, to points f (P1) and f (P2) the* same distance apart. Thus, an isometry is a function *f with the property* *f (P1) f (P2)| = |P1P2|* for any two points *P1, P2. Intuitively speaking, an isometry "moves the* plane rigidly" because it preserves the distance between points. There are many isometries of the plane, but they can be divided into a few simple and obvious types. We show examples of each type below, and, in the next section, we explain why only these types exist.

- You will notice that certain isometries (translations and rotations) make it possible to move the origin to any point in the plane and the *x-axis to any* line. Thus, R^2 is really like Euclid's plane, in the sense that each point is like any other point and each line is like any other line. This property entitles us to choose axes wherever it is convenient. For example, we are entitled to prove the triangle inequality, as suggested in the Exercises to Section 3.3, by choosing one vertex of the triangle at *0 and another on the* positive *x-axis.*

Isometries

Translations

- A translation moves each point of the plane the same distance in the same direction. Each translation depends on two constants *a and b, so we denote* it by *ta,b. It sends each point (x, y) to the point (x+a, y+b). It is obvious* that a translation preserves the distance between any two points, but it is worth checking this formally—so as to know what to do in less obvious cases.

- So let *P1 = (x1, y1) and P2 = (x2, y2). It follows that*

$$t_{a,b}(P_1) = (x_1 + a, y_1 + b), \quad t_{a,b}(P_2) = (x_2 + a, y_2 + b)$$

and therefore,

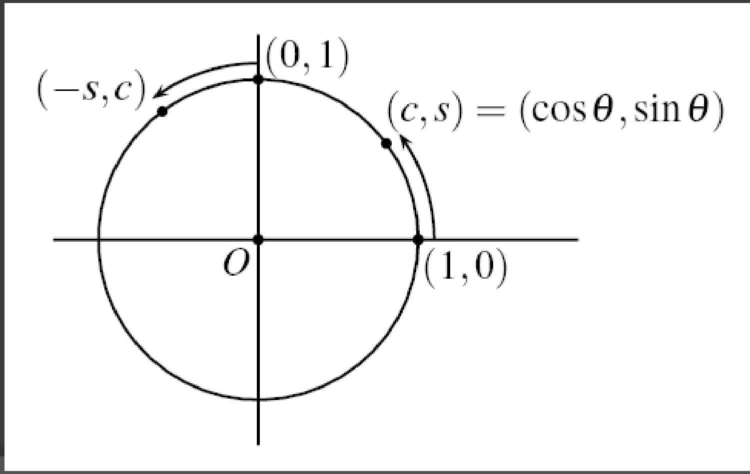
$$\begin{aligned} |t_{a,b}(P_1)t_{a,b}(P_2)| &= \sqrt{(x_2 + a - x_1 - a)^2 + (y_2 + b - y_1 - b)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= |P_1P_2|, \quad \text{as required.} \end{aligned}$$

Isometries

Rotations

- We think of a rotation as something involving an angle θ , but, as mentioned in the previous section, it is more convenient to work algebraically with cosθ and sinθ . These are simply two numbers *c and s such that* $c^2+s^2=1$, so we will denote a rotation of the plane about the origin by *rc,s.*

- The rotation *rc,s sends the point (x, y) to the point (cx− sy, sx+cy). It* is not obvious why this transformation should be called a rotation, but it becomes clearer after we check that *rc,s preserves lengths.* Also, *rc,s sends (0,0) to itself, and it moves* (1,0) to (*c, s) and (0,1) to (− s, c), which is exactly what rotation about 0 through angle θ does. We will see in the next section that* only one isometry of the plane moves these three points in this manner.



Movement of points by a rotation

Isometries

Reflections

- The easiest reflection to describe is *reflection in the x-axis, which sends P = (x, y) to P = (x,−y). Again it is obvious that this is an isometry, but* we can check by calculating the distance between reflected points *P1 and P2.*

- We can reflect the plane in any line, and we can do this by combining reflection in the *x-axis with translations and rotations. For example, reflection* in the line *y = 1 (which is parallel to the x-axis) is the result of the* following three isometries:
  - • *t0,−1, a translation that moves the line y = 1 to the x-axis,*
  - • reflection in the *x-axis,*
  - • *t0,1, which moves the x-axis back to the line y = 1.*

- In general, we can do a reflection in any line L by moving L to the *x-axis* by some combination of translation and rotation, reflecting in the *x-axis,* and then moving the *x-axis back to L.*

- Reflections are the most fundamental isometries, because any isometry is a combination of them, as we will see in the next section. In particular, any translation is a combination of two reflections, and any rotation is a combination of two reflections.

Isometries

Glide Reflections

- A glide reflection is the result of a reflection followed by a translation in the direction of the line of reflection. For example, if we reflect in the *x-axis,* sending *(x, y) to (x,− y), and follow this with the translation t1,0 of length* 1 in the *x-direction, then (x, y) ends up at (x+1,− y).*

- A glide reflection with nonzero translation length is different from the three types of isometry previously considered.
  - It is not a translation, because a translation maps any line in the direction of translation into itself, whereas a glide reflection maps only one line into itself (namely, the line of reflection).
  - It is not a rotation, because a rotation has a fixed point and a glide reflection does not.
  - It is not a reflection, because a reflection also has fixed points (all points on the line of reflection).

The three reflections theorem

- Three reflections theorem. *Any isometry of R2 is a combination of one, two, or three reflections.*

- One reflection is a reflection, and we found in the previous exercise set that combinations of two reflections are translations and rotations, and that combinations of three reflections are glide reflections (which include reflections). Thus, *an isometry of R2 is either a translation, a rotation, or a glide reflection.*

Endnotes for "Coordinates"

- The discovery of coordinates is rightly considered a turning point in the development of mathematics because it reveals a vast new panorama of geometry, open to exploration in at least three different directions.
  - Description of curves by equations, and their analysis by algebra. This direction is called *algebraic geometry, and the curves described* by polynomial equations are called *algebraic curves. Straight lines,* described by the linear equations *ax+by+c=0, are called curves of degree 1. Circles, described by the equations (x– a)2+(y– b)2 =r2,* are curves of *degree 2, and so on.* One can see that there are curves of arbitrarily high degree.
  - Algebraic study of objects described by linear equations (such as lines and planes). Even this is a big subject, called *linear algebra.* Although it is technically part of algebraic geometry, it has a special flavor, very close to that of Euclidean geometry. The real strength of linear algebra is its ability to describe spaces of any number of dimensions in geometric language.
  - The study of transformations, which draws on the special branch of algebra known as *group theory. Because many geometric transformations* are described by linear equations, this study overlaps with linear algebra.

Understanding Geometry through Linear Algebra

Vector and Euclidean spaces

- In this chapter, we process coordinates by *linear algebra. We view* points as *vectors that can be added and multiplied by numbers, and* we introduce the *inner product of vectors, which gives an efficient* algebraic method to deal with both lengths and angles.

- We revisit some theorems of Euclid to see where they fit in the world of vector geometry, and we become acquainted with some theorems that are particularly natural in this environment.

Vector

- Vectors are mathematical objects that can be added, and multiplied by numbers, subject to certain rules. The real numbers are the simplest example of vectors, and the rules for sums and multiples of any vectors are just the following properties of sums and multiples of numbers:

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} \qquad 1\mathbf{u}=\mathbf{u}$$
$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w} \qquad a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$$
$$\mathbf{u}+\mathbf{0}=\mathbf{u} \qquad (a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$$
$$\mathbf{u}+(-\mathbf{u})=\mathbf{0} \qquad a(b\mathbf{u})=(ab)\mathbf{u}.$$

- These rules obviously hold when *a,b,1,u,v,w,0 are all numbers, and 0 is* the ordinary zero.

- They also hold when u,v,w are *points in the plane* R^2, *if we interpret* 0 as (0,0), + as the *vector sum defined for u = (u1,u2) and v = (v1, v2) by*

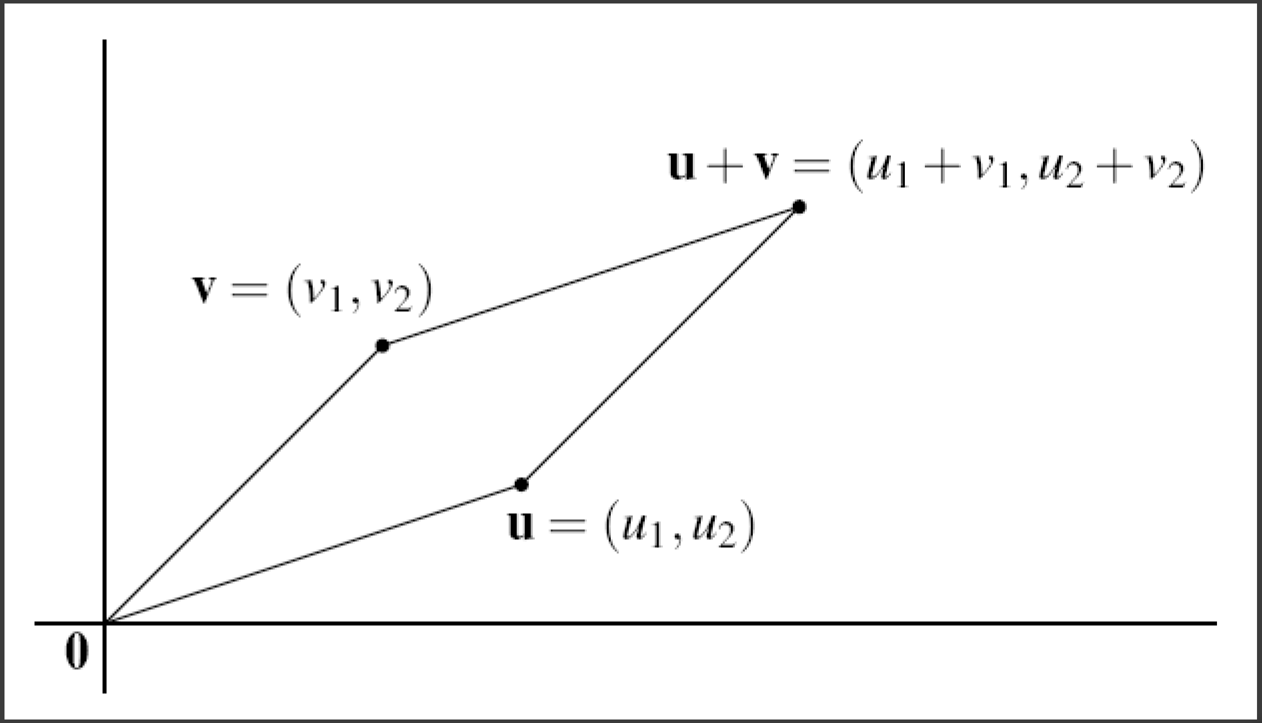
(u1,u2)+(v1+v2) = (u1+v1,u2+v2),

- and *au as the scalar multiple defined by*

a(u1,u2) = (au1,au2).

Vector

- The vector sum is geometrically interesting, because u+v is the fourth vertex of a parallelogram formed by the points 0, u, and v.

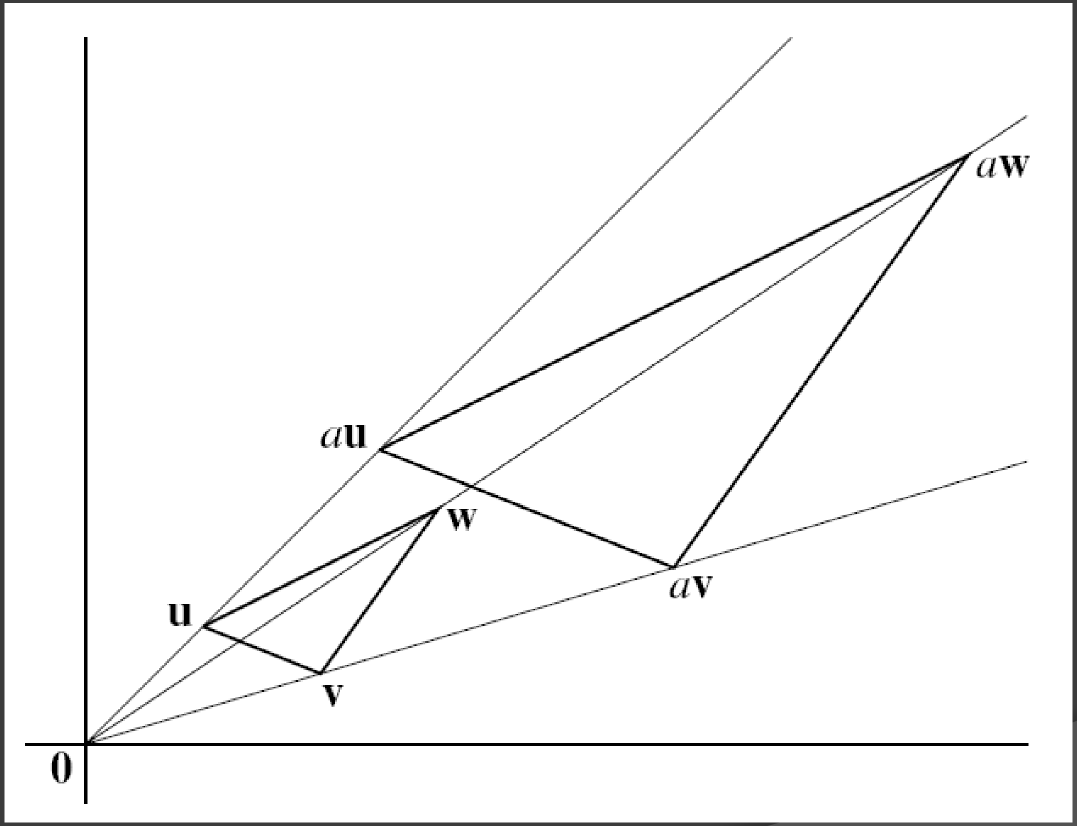


The parallelogram rule for vector sum

- In fact, the rule for forming the sum of two vectors is often called the "parallelogram rule."

Vector

- Scalar multiplication by *a is also geometrically interesting, because it* represents magnification by the factor *a. It magnifies, or dilates, the whole* plane by the factor *a, transforming each figure into a similar copy of itself.*



Scalar multiplication as a dilation of the plane

Real vector spaces

- It seems that the operations of vector addition and scalar multiplication capture some geometrically interesting features of a space. With this in mind, we define a *real vector space to be a set V of objects, called vectors,* with operations of vector addition and scalar multiplication satisfying the following conditions :

  - If u and v are in *V, then so are u+v and au for any real number a.*
  - There is a *zero vector 0 such that u+0 = u for each vector u. Each* u in *V has a additive inverse – u such that u+(– u) = 0.*
  - Vector addition and scalar multiplication on *V have the eight properties* listed at the beginning of this section.

- It turns out that real vector spaces are a natural setting for Euclidean geometry. We must introduce extra structure, which is called the *inner product, before we can talk about length and angle. But once the inner* product is there, we can prove all theorems of Euclidean geometry, often more efficiently than before. Also, we can uniformly extend geometry to *any number of dimensions by considering the space R^n of ordered n-tuples* of real numbers (*x1, x2, . . . , xn).*

Direction and linear independence

- Vectors give a concept of *direction in* R^2 *by representing lines through 0.* If u is a nonzero vector, then the real multiples *au of u make up the line* through 0 and u, so we call them the points "in direction u from 0." (You may prefer to say that –u is in the direction *opposite to u, but it is simpler* to associate direction with a whole line, rather than a half line.)

- Nonzero vectors u and v, therefore, have *different directions from 0 if* neither is a multiple of the other. It follows that such u and v are *linearly independent; that is, there are no real numbers a and b, not both zero,*

  $$\text{with } au+bv = 0.$$

  Because, if one of *a, b is not zero in this equation, we can divide by it and* hence express one of u, v as a multiple of the other.

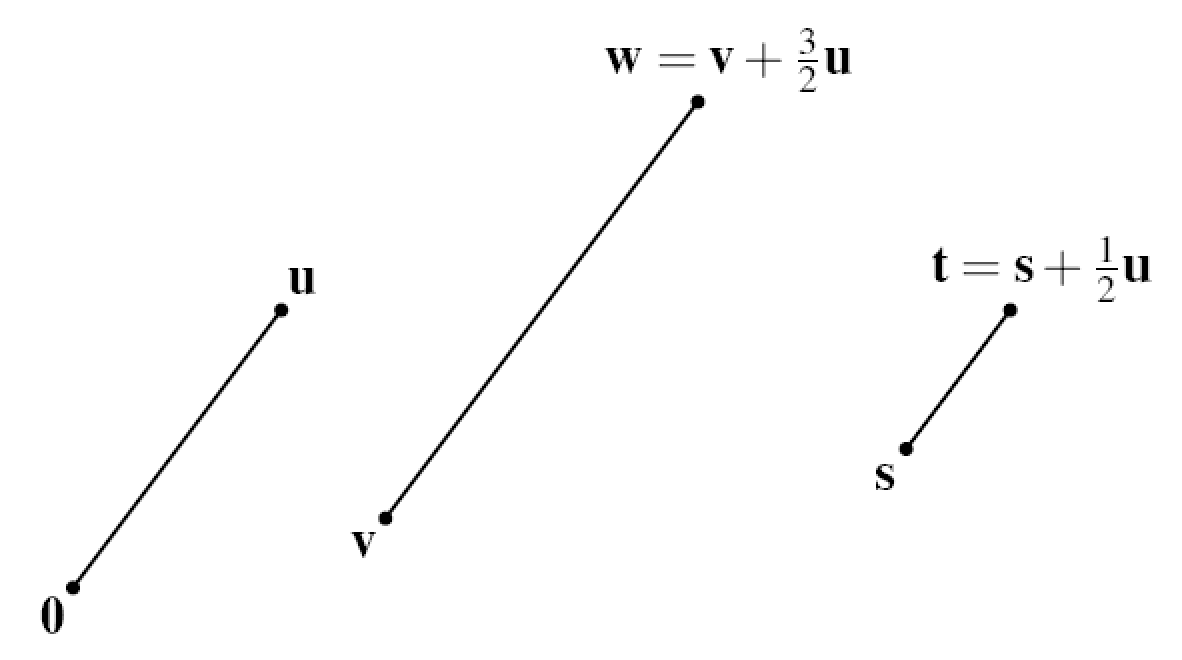
- The concept of direction has an obvious generalization: w *has direction* u *from v (or relative to v) if w– v is a multiple of u. We also say that "w– v* has direction u," and there is no harm in viewing w–v as an abbreviation for the line segment from v to w. As in coordinate geometry, we say that line segments from v to w and from s to t are *parallel if they have the same* direction; that is, if

  $$w - v = a(t - s) \text{ for some real number } a = 0.$$

Direction and linear independence

- Figure below shows an example of parallel line segments, from v to w and from s to t, both of which have direction u. Here we have

$$\mathbf{w}-\mathbf{v}=\frac{3}{2}\mathbf{u} \quad \text{and} \quad \mathbf{t}-\mathbf{s}=\frac{1}{2}\mathbf{u}, \quad \text{so} \quad \mathbf{w}-\mathbf{v}=3(\mathbf{t}-\mathbf{s}).$$



Parallel line segments with direction **u**

Direction and linear independence

The vector concept of parallels on two important theorems.

- **Vector Thales theorem.**

  If s and v are on one line through 0, t and w are on another, and $w - v$ is parallel to $t - s$, then $v = as$ and $w = at$ for some number a.

- **Vector Pappus theorem.**

  If r, s, t, u, v, w lie alternately on two lines through 0, with $u - v$ parallel to $s - r$ and $t - s$ parallel to $v - w$, then $u - t$ is parallel to $w - r$.

Midpoints and centroids

- The definition of a real vector space does not include a definition of distance, but we can speak of the midpoint of the line segment from u to v and, more generally, of the point that divides this segment in a given ratio

- To see why, first observe that v is obtained from u by adding v−u, the vector that represents the position of v *relative to u. More generally,* adding any scalar multiple *a(v− u) to u produces a point whose direction* relative to u is the same as that of v. Thus, the points u+*a(v− u) are* precisely those on the line through u and v. In particular, the midpoint of the segment between u and v is obtained by adding 1 2 (v−u) to u, and hence,

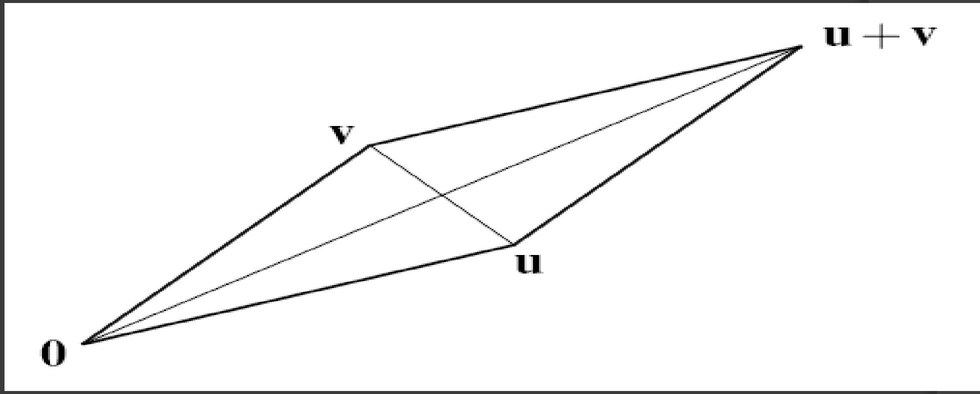
$$\text{midpoint of line segment between } \mathbf{u} \text{ and } \mathbf{v} = \mathbf{u} + \frac{1}{2}(\mathbf{v} - \mathbf{u}) = \frac{1}{2}(\mathbf{u} + \mathbf{v}).$$

One might describe this result by saying that the midpoint of the line segment between u and v is the *vector average of u and v.*

Midpoints and centroids

- This description of the midpoint gives a very short proof of the theorem from that the diagonals of a parallelogram bisect each other.

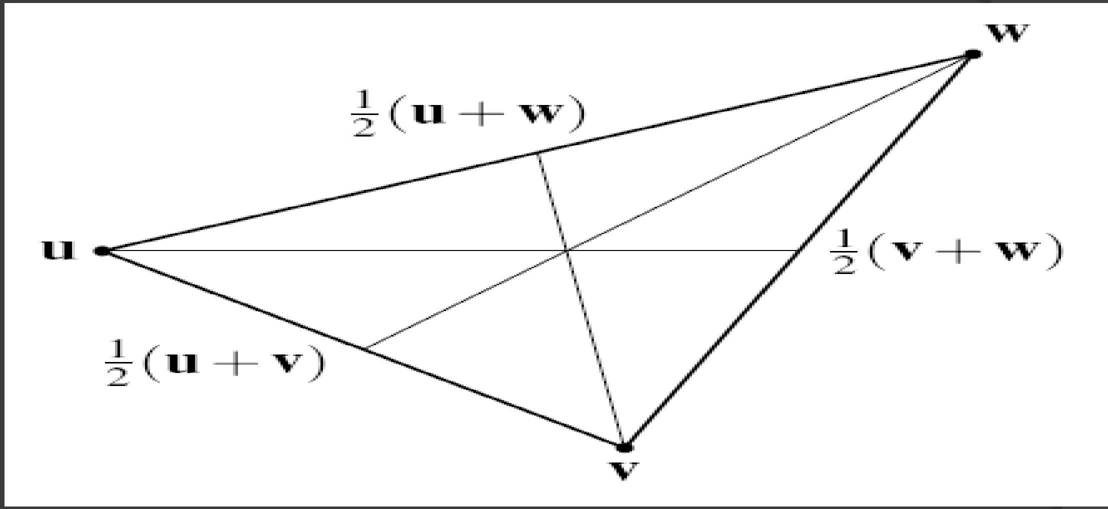
Diagonals of a parallelogram



- Then the midpoint of the diagonal from 0 to u+v is ½ (u+v). And, by the result just proved, this is also the midpoint of the other diagonal—the line segment between u and v.
- The vector average of two or more points is physically significant because it is the *barycenter or center of mass of the system obtained by placing* equal masses at the given points. The geometric name for this vector average point is the *centroid.*
- In the case of a triangle, the centroid has an alternative geometric description, given by the following classical theorem about *medians: the lines* from the vertices of a triangle to the midpoints of the respective opposite sides.

Midpoints and centroids

- **Concurrence of medians.** *The medians of any triangle pass through the same point, the centroid of the triangle.*



The medians of a triangle

- Looking at this figure, it seems likely that the medians meet at the point 2/3 of the way from u to ½ (v+w), that is, at the point
- This is the centroid, and a similar argument shows that it lies 2/3 of the way between v and ½ (u+w) and 2/3 of the way between w and ½ (u+v). That is, the centroid is the common point of all three medians.

The inner product

- If u = (*u1,u2) and v = (v1, v2) are vectors in R^2, we define their inner product u · v to be u1v1 + u2v2. Thus, the inner product of two vectors is* not another vector, but a real number or "scalar." For this reason, u · v is also called the *scalar product of u and v.*

- It is easy to check, from the definition, that the inner product has the algebraic properties

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u},\\
\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w},\\
(a\mathbf{u})\cdot\mathbf{v} &= \mathbf{u}\cdot(a\mathbf{v}) = a(\mathbf{u}\cdot\mathbf{v}),
\end{aligned}
$$

which immediately give information about length and angle:

- The length |u| is the distance of u=(*u1,u2) from 0,* by the definition of distance in R^2
- Vectors u and v are perpendicular if and only if u · v = 0

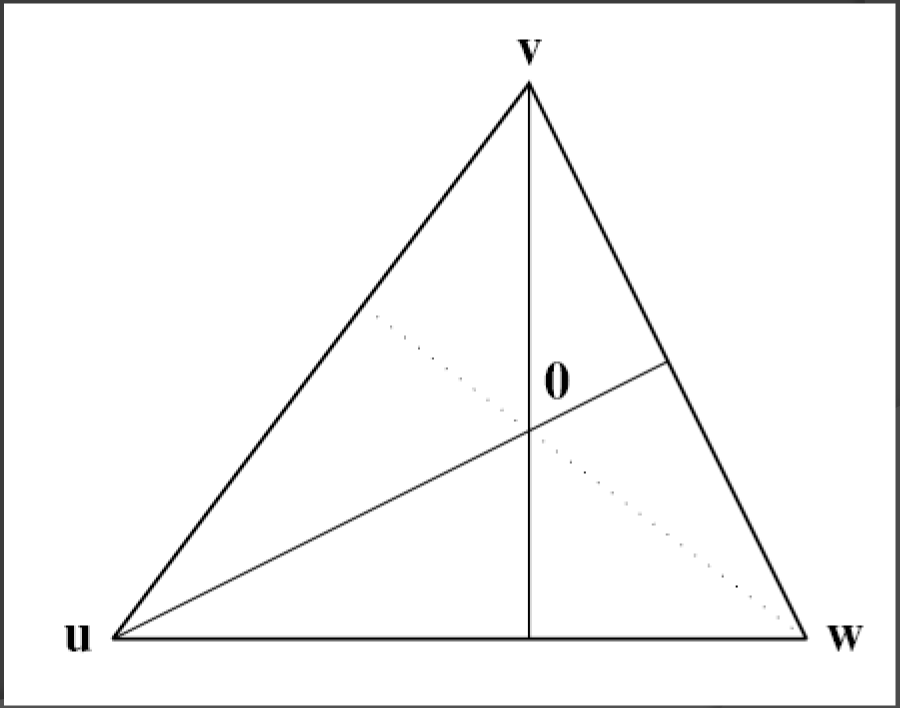
The inner product

Concurrence of altitudes.

In any triangle, the perpendiculars from the vertices to opposite sides (the altitudes) have a common point.

- **To prove this theorem, take 0 at the intersection of two altitudes, say those through the vertices u and v. Then it remains to show that the line from 0 to the third vertex w is perpendicular to the side v–u.**

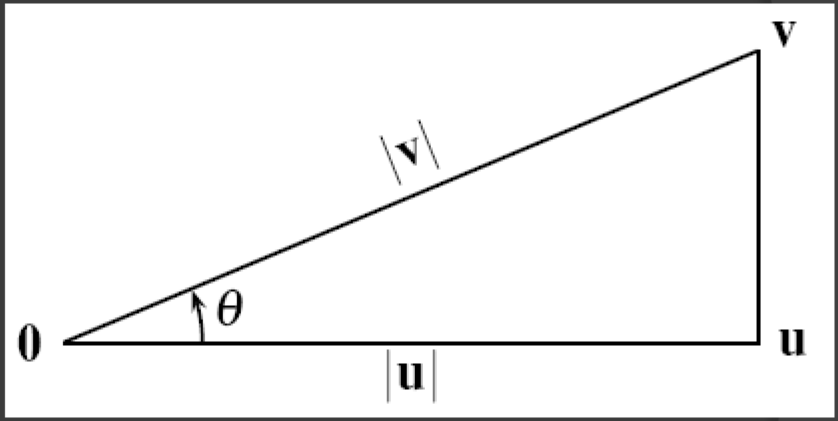
Altitudes of a triangle



The inner product and cosine

- The inner product of vectors u and v depends not only on their lengths |u| and |v| but also on the angle θ between them. The simplest way to express its dependence on angle is with the help of the *cosine function. We* write the cosine as a function of angle θ , cosθ . But, as usual, we avoid measuring angles and instead define cosθ as the ratio of sides of a rightangled triangle. For simplicity, we assume that the triangle has vertices 0, u, and v as shown in the figure below.

Cosine as a ratio of lengths



- Then the side v is the hypotenuse, θ is the angle between the side u and the hypotenuse, and its cosine is defined by

$$\cos\theta = \frac{|\mathbf{u}|}{|\mathbf{v}|}.$$

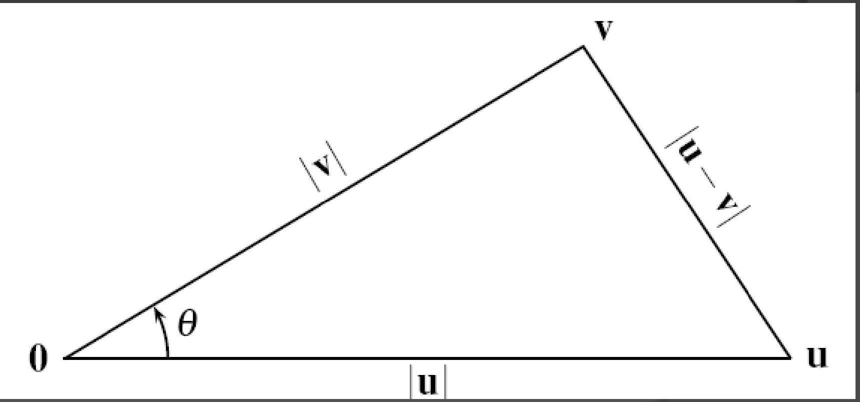
The inner product and cosine

- **Inner product formula.** *If θ Is the angle between vectors u and v, then*
  $u \cdot v = |u||v|\cos\theta$.

  This formula gives a convenient way to calculate the angle (or at least its cosine) between any two lines, because we know how to calculate |u| and |v|. It also gives us the "cosine rule" of trigonometry directly from the calculation of (u–v) ·(u–v).

- **Cosine rule.** *In any triangle, with sides u, v, and u– v, and angle θ opposite to the side u– v,*
  $|u-v|2 = |u|2+|v|2-2|u||v|\cos\theta$.

Quantities mentioned in the cosine rule



The triangle inequality

- In vector geometry, the triangle inequality |u+v| ≤ |u|+|v| is usually derived from the fact that |u·v| ≤ |u||v|. This result, known as the *Cauchy–Schwarz inequality*.

- The reason for the fuss about the Cauchy–Schwarz inequality is that it holds in spaces more complicated than R2, with more complicated inner products. Because the triangle inequality follows from Cauchy–Schwarz, it too holds in these complicated spaces. We are mainly concerned with the geometry of the plane, so we do not need complicated spaces. However, it is worth saying a few words about R*n, because linear algebra works just as* well there as it does in R2.

- Higher dimensional Euclidean spaces

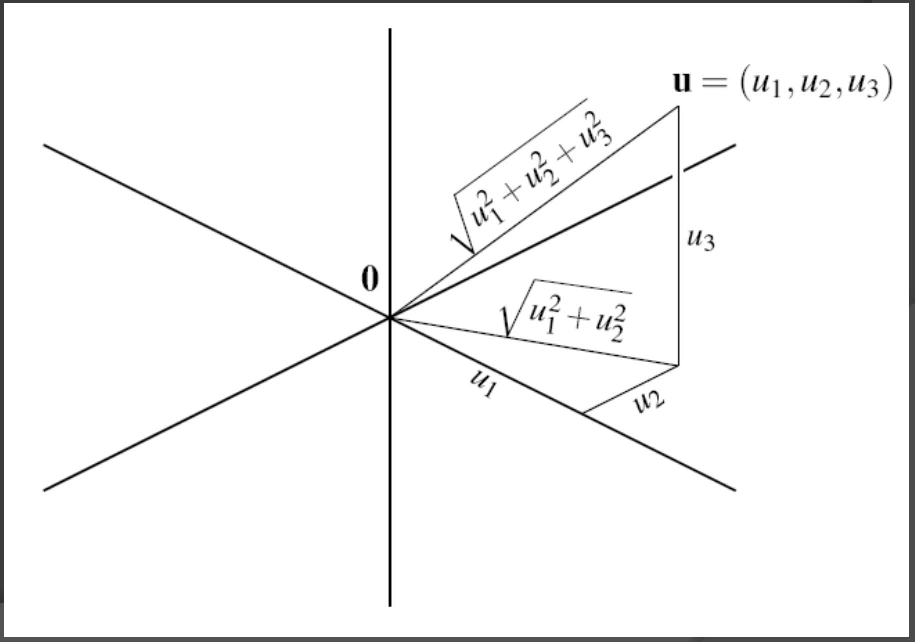
Higher dimensional Euclidean spaces

- R^n *is the set of ordered n-tuples (x1, x2, . . . ,xn) of real numbers x1, x2, . . . ,xn.* These ordered *n-tuples are called n-dimensional vectors.*

- It is easy to check that R^n *has the properties enumerated previously.* Hence, R^n *is a real vector space under the vector sum and* scalar multiplication operations just described.

- R^n *becomes a Euclidean space when we give it the extra structure of* an inner product with the properties enumerated previously. For example, the distance of (*u1,u2,u3) from 0 in R^3 is*

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2},$$



Distance in R^3

Higher dimensional Euclidean spaces

- All theorems proved in this chapter for vectors in the plane R^2 hold in R^n. *This fact is clear if we take the plane in Rn to consist of vectors of* the form (*x1, x2,0, . . . ,0), because such vectors behave exactly the same as* vectors (*x1, x2) in R^2. But in fact any given plane in Rn behaves the same* as the special plane of vectors (*x1, x2,0, . . . ,0). We skip the details, but it* can be proved by constructing an isometry of R^n *mapping the given plane* onto the special plane. As in R^2, any isometry is a product of reflections. In R^n, *at most n+1 reflections are required.*

Rotations, matrices, and complex numbers

Rotation matrices

- We have defined a rotation of $\mathbb{R}^2$ as a function *$r_{c,s}$, where c and s* are two real numbers such that *$c^2+s^2=1$. We described $r_{c,s}$ as the function* that sends *(x, y) to (cx− sy, sx+cy), but it is also described by the matrix of coefficients of x and y, namely*

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad \text{where } c = \cos\theta \text{ and } s = \sin\theta.$$

- Matrix notation allows us to rewrite *(x, y)→(cx− sy, sx+cy) as*

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx - sy \\ sx + cy \end{pmatrix}$$

- Functions are thereby separated from their variables, so they can be composed without the variables becoming involved—simply by multiplying matrices.

Rotations, matrices, and complex numbers

Rotation matrices

- This idea gives proofs of the formulas for cos(θ1+θ2) and sin(θ1+θ2), but with the variables *x and y filtered* out :
  - Rotation through angle θ1 is given by the matrix

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

  - Rotation through angle θ2 is given by the matrix

$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}.$$

  - Rotation through θ1+θ2 is given by the product of these two matrices. That is by matrix multiplication
  - Finally, equating corresponding entries in the first and last matrices,

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$
$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.$$

Rotations, matrices, and complex numbers

Rotation matrices

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$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}.$$

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  - Finally, equating corresponding entries in the first and last matrices,

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$$\sin(\theta_1+\theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.$$

Rotations, matrices, and complex numbers

Complex numbers

- One advantage of matrices, which we do not pursue here, is that they can be used to generalize the idea of rotation to any number of dimensions. But, for rotations of R^2, there is a notation even more efficient than the rotation matrix.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

It is the *complex number* $\cos\theta + i\sin\theta$*, where* $i = \sqrt{-1}$*.*

- We represent the point $(x, y) \in R^2$ *by the complex number* $z = x+iy$, and we rotate it through angle θ about *0 by multiplying it by* $\cos\theta + i\sin\theta$. This procedure works because $i^2 = -1$*, and therefore,* $(\cos\theta + i\sin\theta)(x+iy) = x\cos\theta - y\sin\theta + I(x\sin\theta + y\cos\theta)$.

- Thus, multiplication by $\cos\theta + i\sin\theta$ *sends each point (x, y) to the point* $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$*, which is the result of rotating (x, y) about 0 through angle* θ*. Multiplying all points at once by* $\cos\theta + i\sin\theta$*, therefore,* rotates the *whole plane about 0 through angle* θ*.*

Endnotes for "Vector & Eucledian Spaces"

- Because the geometric content of a vector space with an inner product is much the same as Euclidean geometry, it is interesting to see how many axioms it takes to describe a vector space.
- To define a vector space, we began with eight axioms for vector addition and scalar multiplication:

$$
\begin{aligned}
\mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} & 1\mathbf{u} &= \mathbf{u} \\
\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} & a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u}+a\mathbf{v} \\
\mathbf{u}+\mathbf{0} &= \mathbf{u} & (a+b)\mathbf{u} &= a\mathbf{u}+b\mathbf{u} \\
\mathbf{u}+(-\mathbf{u}) &= \mathbf{0} & a(b\mathbf{u}) &= (ab)\mathbf{u}.
\end{aligned}
$$

- Then, we added three (or four, depending on how you count) axioms for the inner product :

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u}, \\
\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}, \\
(a\mathbf{u})\cdot\mathbf{v} &= \mathbf{u}\cdot(a\mathbf{v}) = a(\mathbf{u}\cdot\mathbf{v}),
\end{aligned}
$$

Endnotes for "Vector & Eucledian Spaces"

- We also need relations among inner product, length, and angle—at a minimum the cosine formula,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta,$$

- At the very least, one needs axioms saying that the scalars satisfy the ordinary rules of calculation, the so-called *field axioms (this is usual when defining a vector* space) :

$$\begin{array}{ccr} a+b=b+a, & ab=ba & \text{(commutative laws)} \\ a+(b+c)=(a+b)+c, & a(bc)=(ab)c & \text{(associative laws)} \\ a+0=a, & a1=a & \text{(identity laws)} \\ a+(-a)=0, & aa^{-1}=1 & \text{(inverse laws)} \\ & a(b+c)=ab+ac & \text{(distributive law)} \end{array}$$

- Thus, the usual definition of a vector space, with an inner product suitable for Euclidean geometry, takes more than 20 axioms! Admittedly, the field axioms and the vector space axioms are useful in many other parts of mathematics, whereas most of the Hilbert axioms seem meaningful only in geometry. And, by varying the inner product slightly, one can change the geometry of the vector space in interesting ways.

Endnotes for "Vector & Eucledian Spaces"

- Still, one can dream of building geometry on a much simpler set of axioms.
- In the next section, we will realize this dream with projective geometry.