EMPAT PENDEKATAN DALAM PEMAHAMAN TENTANG GEOMETRI

Disarikan dari :

THE FOUR PILLARS OF GEOMETRY

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INTRODUCTION

- Geometry can be developed in four fundamentally different ways, and that *all* should be used if the subject is to be shown in all its splendor.
  - *Euclid-style construction and axiomatics*
  - *linear algebra*
  - *projective geometry*
  - *transformation groups*

- Geometry, of all subjects, should be about *taking different viewpoints*, and geometry is unique among the mathematical disciplines in its ability to look different from different angles. Some prefer to approach it visually, others algebraically, but the miracle is that they are all looking at the same thing.

EUCLID-STYLE CONSTRUCTION AND AXIOMATICS

Straightedge and compass

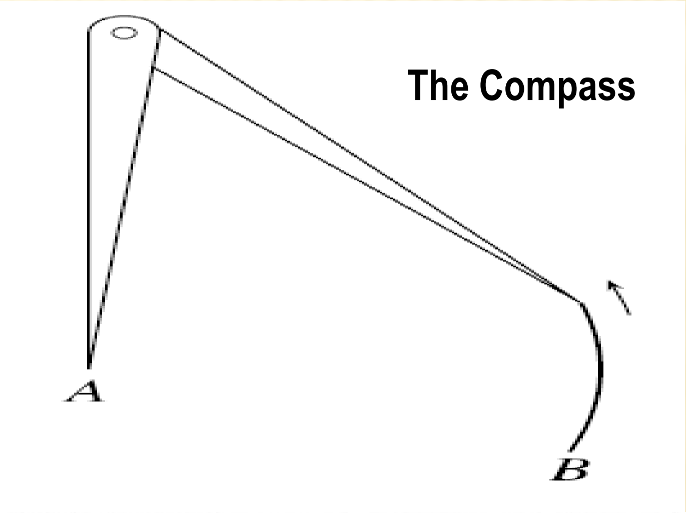
- For over 2000 years, mathematics was almost synonymous with the geometry of Euclid's Elements, a book written around 300 BCE and used in school mathematics instruction until the 20th century. *Euclidean geometry*, as it is now called, was thought to be the foundation of all exact science

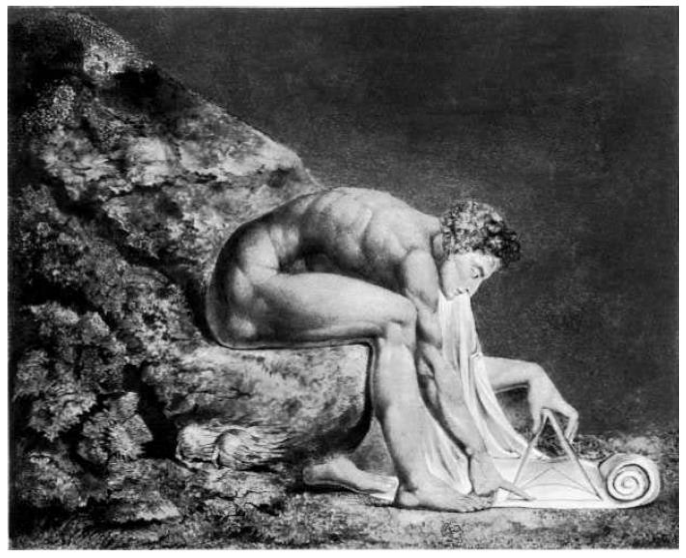
- A naive way to describe Euclidean geometry is to say it concerns the geometric figures that can be drawn (or *constructed* as we say) by straightedge and compass. Euclid assumes that it is possible to draw a straight line between any two given points, and to draw a circle with given center and radius. All of the propositions he proves are about figures built from straight lines and circles. Thus, to understand Euclidean geometry, one needs some idea of the scope of straightedge and compass constructions.

Euclid's construction axioms

- Euclid assumes that certain constructions can be done and he states these assumptions in a list called his *axioms* (traditionally called *postulates*). He assumes that it is possible to:
  1. Draw a straight line segment between any two points.
  2. Extend a straight line segment indefinitely.
  3. Draw a circle with given center and radius.

- Axioms 1 and 2 say we have a *straightedge*, an instrument for drawing arbitrarily long line segments. Today we replace Axioms 1 and 2 by the single axiom that a *line* can be drawn through any two points. The straightedge (unlike a ruler) has no scale marked on it and hence can be used *only* for drawing lines—not for measurement. Euclid separates the function of measurement from the function of drawing straight lines by giving measurement functionality only to the *compass*—the instrument assumed in Axiom 3. The compass is used to draw the circle through a given point *B*, with a given point *A* as center (Figure 1.1).





Blake's painting of Newton the measurer

Euclid's construction axioms

- The compass also enables us to *add* and *subtract* the length |*AB*| of *AB* from the length |*CD*| of another line segment *CD* by picking up the compass with radius set to |*AB*| and describing a circle with center *D*. By adding a fixed length repeatedly, one can construct a "scale" on a given line, effectively creating a ruler. This process illustrates how the power of measuring lengths resides in the compass. Exactly which lengths can be measured in this way is a deep question, which belongs to algebra and analysis.
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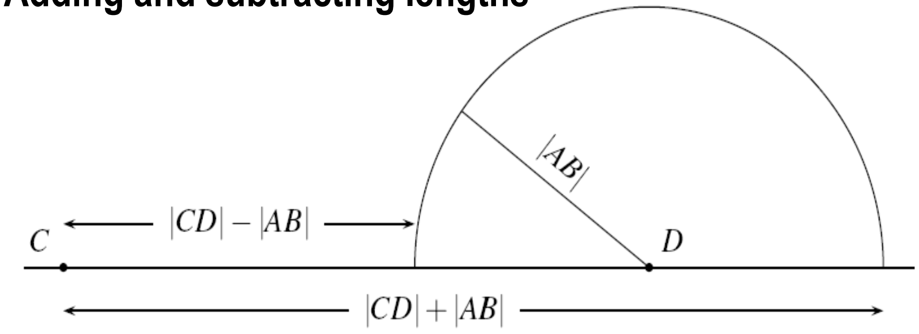
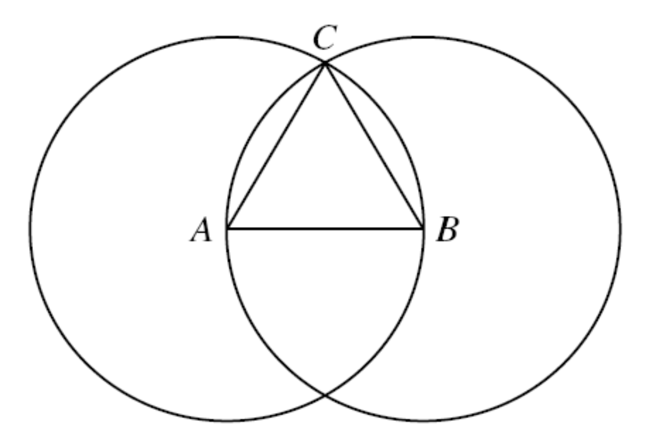
- Separating the concepts of "straightness" and "length," as the straightedge and the compass do, turns out to be important for understanding the foundations of geometry.

Adding and subtracting lengths



Euclid's construction of the equilateral triangle

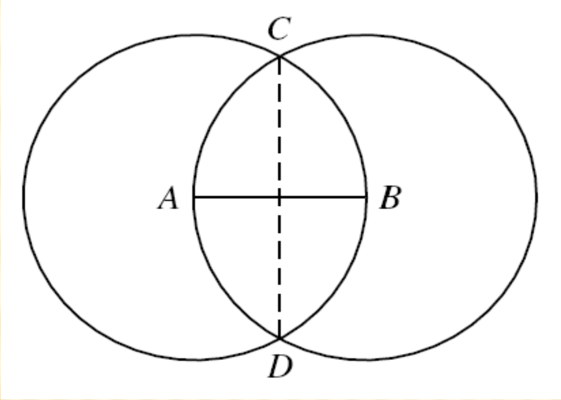
- Constructing an equilateral triangle on a given side *AB* is the first proposition of the *Elements*, and it takes three steps:
    1. *Draw the circle with center A and radius AB.*
    2. *Draw the circle with center B and radius AB.*
    3. *Draw the line segments from A and B to the intersection C of the two circles just constructed.*
- The result is the triangle *ABC* with sides *AB*, *BC*, and *CA* in Figure 1.4. Sides *AB* and *CA* have equal length because they are both radii of the first circle. Sides *AB* and *BC* have equal length because they are both radii of the second circle. Hence, all three sides of triangle *ABC* are equal.



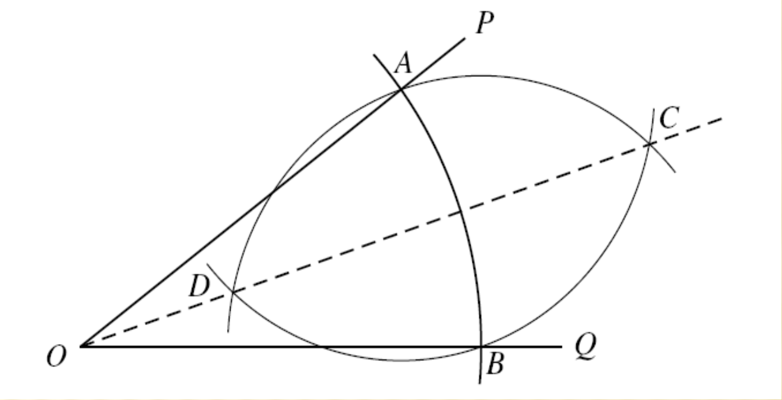
Constructing an equilateral triangle

Some basic constructions

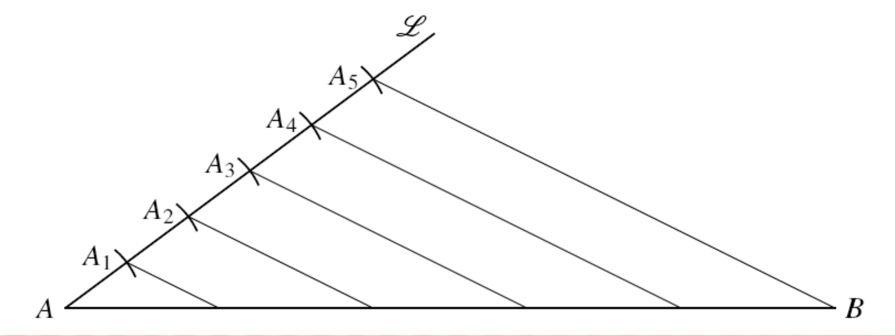
- The equilateral triangle construction comes first in the *Elements* because several other constructions follow from it. Among them are constructions for bisecting a line segment and bisecting an angle. ("Bisect" is from the Latin for "cut in two.")



Bisecting a line segment *AB*



Bisecting an angle *POQ*

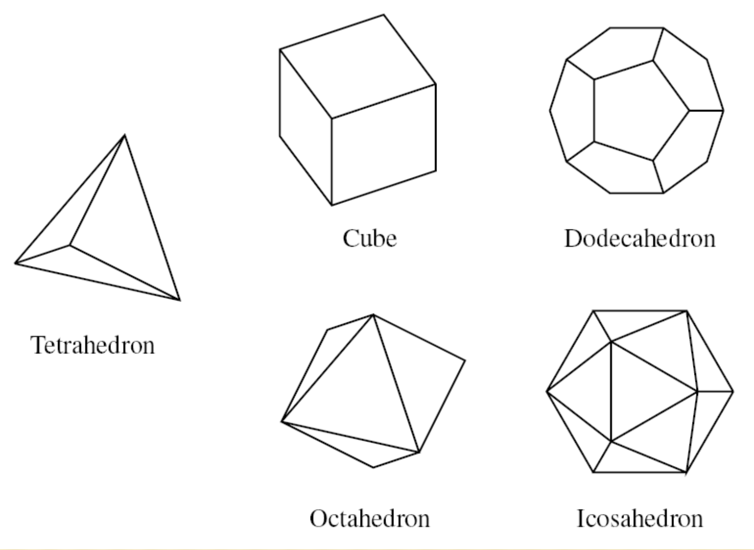


Dividing a line segment into equal parts

EUCLID'S "ELEMENTS"

- Euclid's Elements is the most influential book in the history of mathematics. The climax of the Elements is the theory of regular polyhedra. Only five regular polyhedra exist. Three of them are built from equilateral triangles, one from squares, and one from regular pentagons. This remarkable phenomenon underlines the importance of equilateral triangles and squares, and draws attention to the regular pentagon. Some geometers believe that the material in the Elements was chosen very much with the theory of regular polyhedra in mind. For example, Euclid wants to construct the equilateral triangle, square, and pentagon in order to construct the regular polyhedra.

The regular polyhedra

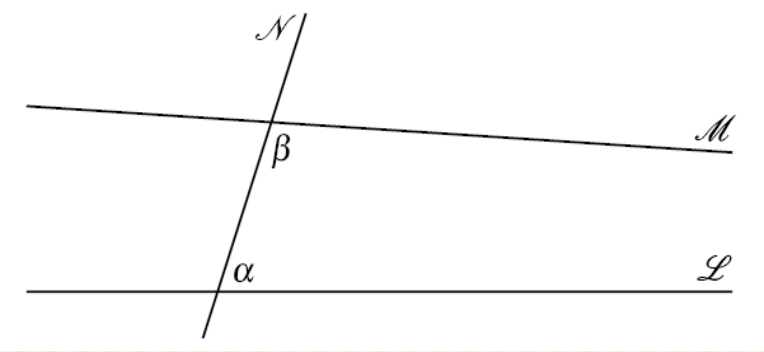


EUCLID'S APPROACH TO GEOMETRY

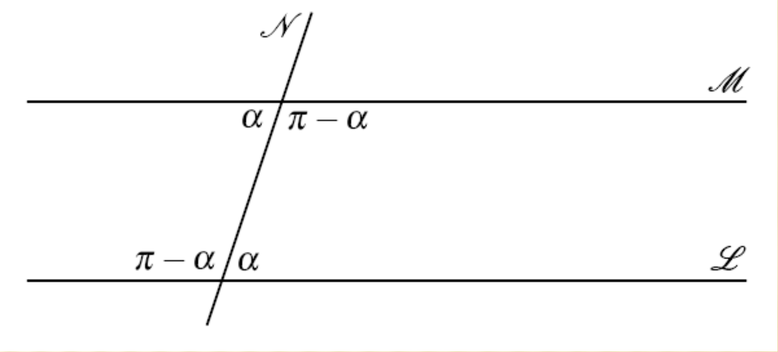
- Length is the fundamental concept of Euclid's geometry, but several important theorems seem to be "really" about angle or area—for example, the theorem on the sum of angles in a triangle and the Pythagorean theorem on the sum of squares. Also, Euclid often uses area to prove theorems about length, such as the Thales theorem.
- In this chapter, we retrace some of Euclid's steps in the theory of angle and area to show how they lead to the Pythagorean theorem and the Thales theorem. We begin with his theory of angle, which shows most clearly the influence of his *parallel axiom*, the defining axiom of what is now called *Euclidean geometry.*
- Angle is linked with length from the beginning by the so-called SAS ("side angle side") criterion for equal triangles (or "congruent triangles," as we now call them). We observe the implications of SAS for isosceles triangles and the properties of angles in a circle, and we note the related criterion, ASA ("angle side angle").
- The theory of area depends on ASA, and it leads directly to a proof of the Pythagorean theorem. It leads more subtly to the Thales theorem and its consequences that we saw in Chapter 1. The theory of angle then combines nicely with the Thales theorem to give a second proof of the Pythagorean theorem.
- In following these deductive threads, we learn more about the scope of straightedge and compass constructions, partly in the exercises. Interesting spinoffs from these investigations include a process for cutting any polygon into pieces that form a square, a construction for the square root of any length, and a construction of the regular pentagon.

The parallel axiom

- **Euclid's parallel axiom.**
  - *If a straight line crossing two straight lines makes the interior angles on one side together less than two right angles, then the two straight lines will meet on that side.*



When lines are not parallel



When lines are parallel

- **Modern parallel axiom.**
  - *For any line* L *and point P outside* L*, there is exactly one line through P that does not meet* L.

- **Angles in a triangle**
  - *The existence of parallels and the equality of alternate interior angles imply a beautiful property of triangles.*
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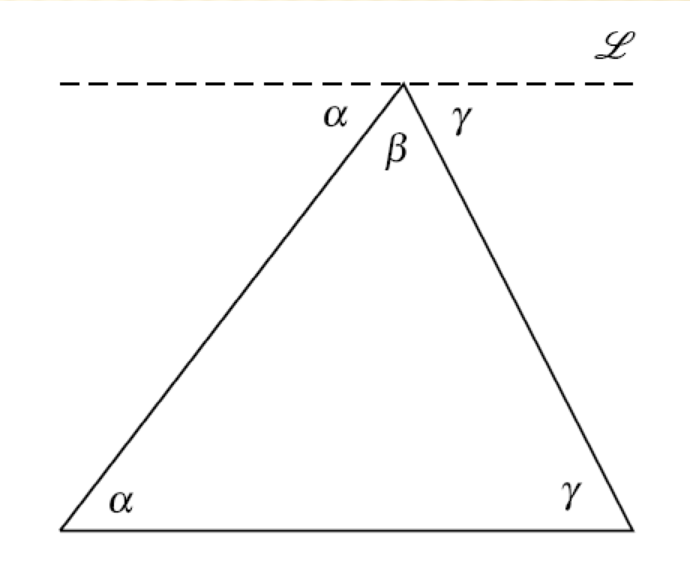
- **Angle sum of a triangle**
  - *If* α*,* β *, and* γ *are the angles of any triangle, then* $\alpha + \beta + \gamma = \pi$.

The angle sum of a triangle



Congruence axioms

- Euclid says that two geometric figures *coincide* when one of them can be moved to fit exactly on the other. He uses the idea of moving one figure to coincide with another in the proof of Proposition 4 of Book I: *If two triangles have two corresponding sides equal, and the angles between these sides equal, then their third sides and the corresponding two angles are also equal.*
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- His proof consists of moving one triangle so that the equal angles of the two triangles coincide, and the equal sides as well. But then the third sides necessarily coincide, because their endpoints do, and hence, so do the other two angles.
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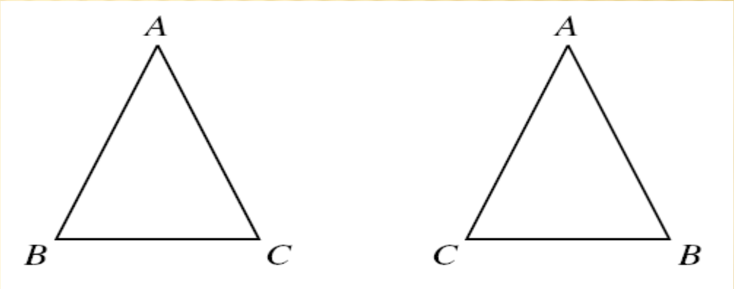
- Today we say that two triangles are *congruent* when their corresponding angles and side lengths are equal, and we no longer attempt to prove the proposition above. Instead, we *take it as an axiom* (that is, an unproved assumption), because it seems simpler to assume it than to introduce the concept of motion into geometry. The axiom is often called SAS (for "side angle side").

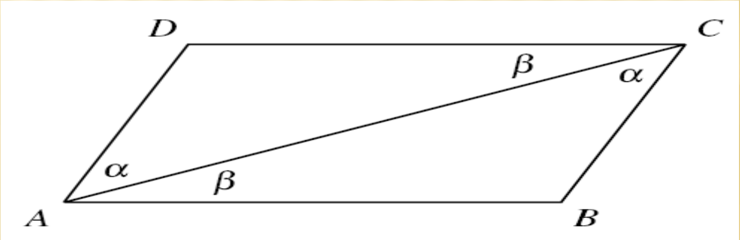
- SAS (side angle side) axiom
  - For brevity, one often expresses SAS by saying that two triangles are congruent if two sides and the included angle are equal.
- Isosceles triangle theorem
  - If a triangle has two equal sides, then the angles opposite to these sides are also equal.

Two views of an isosceles triangle



- Parallelogram side theorem
  - Opposite sides of a parallelogram are equal.

Dividing a parallelogram into triangles



Area and equality

- The principle of logic that “things equal to the same thing are equal to each other” is one of five principles that Euclid calls *common notions*. The common notions he states are particularly important for his theory of area, and they are as follows:
  - 1. Things equal to the same thing are also equal to one another.
  - 2. If equals are added to equals, the wholes are equal.
  - 3. If equals are subtracted from equals, the remainders are equal.
  - 4. Things that coincide with one another are equal to one another.
  - 5. The whole is greater than the part.
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- The word “equal” here means “equal in some specific respect.” In most cases, it means “equal in length” or “equal in area”. Likewise, “addition” can mean addition of lengths or addition of areas, but Euclid never adds a length to an area because this has no meaning in his system.

The square of a sum

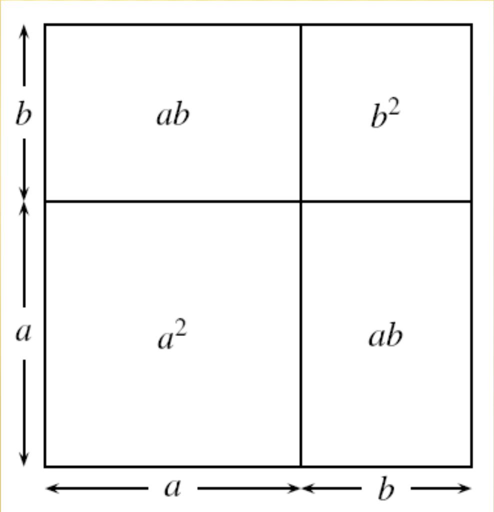
Proposition 4 of Book II is another interesting example. It states a property of squares and rectangles that we express by the algebraic formula

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Euclid does *not* have algebraic notation, so he has to state this equation in words: *If a line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.* Whichever way you say it, Figure 2.8 explains why it is true.

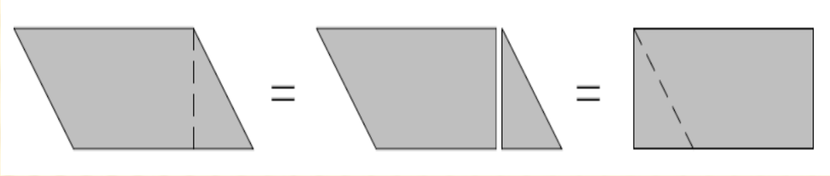
The square of a sum of line segments

- The square on the line is what we write as (*a*+*b*)2.
- The squares on the two segments *a* and *b* are *a*2 and *b*2, respectively.
- The rectangle “contained” by the segments *a* and *b* is *ab*.
- The square (*a*+*b*)2 equals (in area) the sum of *a*2, *b*2, and two copies of *ab*.

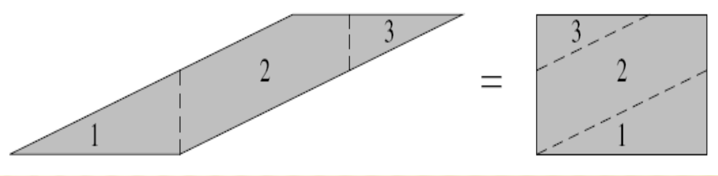


Area of parallelograms and triangles

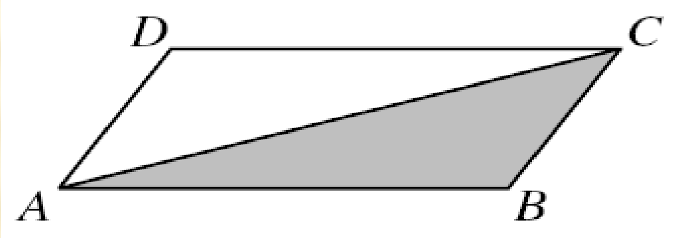
- The first nonrectangular region that can be shown “equal” to a rectangle in Euclid’s sense is a parallelogram.



Assembling parallelogram and rectangle from the same pieces



A case in which more cuts are required



A triangle as half a parallelogram

$$\text{area of triangle} = \frac{1}{2}\,\text{base} \times \text{height}.$$

This formula is important in two ways:

As a statement about area. From a modern viewpoint, the formula gives the area of the triangle as a product of numbers. From the ancient viewpoint, it gives a rectangle “equal” to the triangle, namely, the rectangle with the same base and half the height of the triangle.

As a statement about proportionality. For triangles with the same height, the formula shows that their areas are proportional to their bases. This statement turns out to be crucial for the proof of the Thales theorem (Section 2.6).

Proof of The Pythagorean Theorem &Thales Theorem

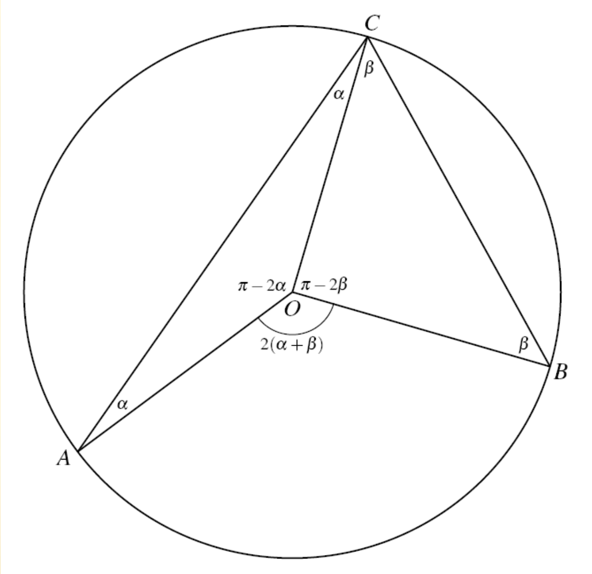
- **By developeing the theory of area for parallelograms and triangles in Book I of the Elements, Euclid could explain the proof of the** ***Pythagorean Theorem &Thales Theorem***

  - **Pythagorean Theorem.** ***For any right-angled triangle, the sum of the squares on the two shorter sides equals the square on the hypotenuse.***

  - **Thales Theorem.** ***A line drawn parallel to one side of a triangle cuts the other two sides proportionally.***

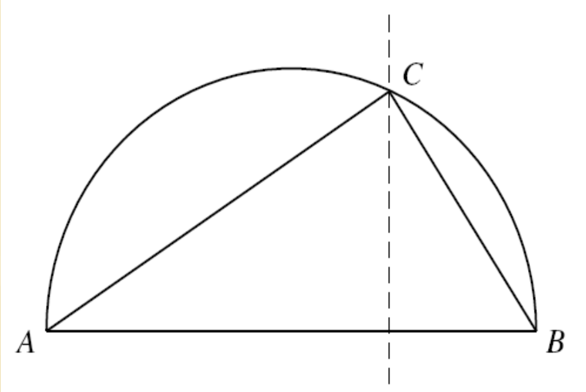
Angles in a circle

- **Invariance of angles in a circle.**
  - *If A and B are two points on a circle, then, for all points C on one of the arcs connecting them, the angle ACB is constant.*



Angle α +β in a circle

- **Angle in a semicircle theorem.**
  - *If A and B are the ends of a diameter of a circle, and C is any other point on the circle, then angle ACB is a right angle.*



Constructing a right-angled triangle with given hypotenuse

Conclusion

- Euclid found the most important axiom of geometry—the parallel axiom—and he also identified the basic theorems and traced the logical connections between them. However, his approach misses certain fine points and is not logically complete. There are many situations, in which Euclid assumes something is true because it *looks* true in the diagram.

- These gaps in Euclid's approach to geometry were first noticed in the 19th century, and the task of filling them was completed by David Hilbert in his "Foundations of Geometry" (1899). The downside of Hilbert's completion of Euclid is that it is lengthy and difficult. Nearly 20 axioms are required, and some key theorems are hard to prove. To some extent, this hardship occurs because Hilbert insists on geometric definitions of + and ×. He wants numbers to come from "inside" geometry rather than from "outside".

- Later, we will take the real numbers as the starting point of geometry, and see what advantages this may have over the Euclid–Hilbert approach. One clear advantage is *access to algebra*, which reduces many geometric problems to simple calculations. Algebra also offers some conceptual advantages, as we will see.